Classical Natural Deduction for S4 Modal Logic

Daisuke Kimura and Yoshihiko Kakutani

Department of Computer Science, University of Tokyo
7-3-1 Hongo, Tokyo 113-8656, Japan
`kimura@lyon.is.s.u-tokyo.ac.jp`, `kakutani@is.s.u-tokyo.ac.jp`

Abstract. This paper proposes a natural deduction system $\mathrm{CND}^{\mathrm{S4}}$ for classical S4 modal logic with necessity and possibility modalities. This new system is an extension of Parigot's Classical Natural Deduction with dual-context to formulate S4 modal logic. The modal $\lambda\mu$-calculus is also introduced as a computational extraction of $\mathrm{CND}^{\mathrm{S4}}$. It is an extension of both the $\lambda\mu$-calculus and the modal λ-calculus. Subject reduction, confluency, and strong normalization of the modal $\lambda\mu$-calculus are shown. Finally, the computational interpretation of the modal $\lambda\mu$-calculus, especially the computational meaning of the modal possibility operator, is discussed.

1 Introduction

Classical Natural Deduction (CND) [16] is a natural deduction system for classical logic. It is introduced to extend the paradigm 'proofs as programs' to classical logic.

Proofs as programs is known as the Curry-Howard correspondence, which is an isomorphism between proofs in logical systems and programs in computational systems. It is studied widely, since it gives computational aspect in logical systems and theoretical foundation of programming languages. The typical example of the correspondence is the one between intuitionistic propositional logic and the simply typed λ-calculus.

Griffin [7] extended the Curry-Howard correspondence to classical logic by discovering the connection between the type of `call/cc` and Peirce's law. The $\lambda\mu$-calculus introduced by Parigot [16] corresponds to CND in the same way that the λ-calculus corresponds to intuitionistic natural deduction. The $\lambda\mu$-calculus has played a central role for studying the Curry-Howard correspondence of classical logic in many approaches such as semantics, abstract machine, functional programming with exception handling, and the computational duality between call-by-value and call-by-name [17,15,2,20,5,8,23,9,11]D.

The Curry-Howard correspondence is also extended to intuitionistic modal logic. Davies and Pfenning [4] showed that the λ-calculus with the S4 modal necessity operator $\Box$ provides a theoretical framework of staged computation by interpreting a formula $\Box A$ as a type of program codes of type A. Staged computation is a computational mechanism that is used for programming techniques such as dynamic code generation and partial evaluation [22]. This mechanism is realized by specifying stages where partial programs should be executed. A partial program that can be used at the current or *any* later stages is treated as a program code at the current stage. Type theoretic approach of staged computation based on intuitionistic modal logic is studied actively [4,3,6,18,13,24].

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The purpose of this paper is the following two points. First, we aim to extend the paradigm 'proofs as programs' to classical modal proofs. We extend CND to classical S4 modal logic, and construct a term calculus that corresponds to the extended system. Second, we aim to give a computational interpretation of classical modal proofs. In particular, we focus on the computational meaning of the modal possibility operator. The λ-calculi with both the modal necessary and possibility operators were introduced in [1,18]. However, since they were constructed based on the analysis from a logical viewpoint, it is still unclear how the possibility operator is interpreted in staged computation.

Some computational systems for classical modal logic have been proposed. Kakutani [10] introduced the $\lambda\mu$-calculi for classical normal modal logic starting from categorical semantics, and extended the computational duality in classical logic to classical modal logic. Shan [21] gave a term calculus that corresponds to sequent calculus for classical S4 modal logic.

This paper presents an extension of classical natural deduction $\mathrm{CND}^{\mathrm{S4}}$ for classical S4 modal logic. This system is a natural deduction system with multiple conclusions to formulate classical logic, and dual-context to formulate S4 modal logic. $\mathrm{CND}^{\mathrm{S4}}$ has both the modal necessity and possibility operators as primitives. We then introduce the $\lambda\mu^{S4}$-calculus as the extracted computational system from $\mathrm{CND}^{\mathrm{S4}}$. It extends proofs as types of classical logic to classical modal logic. The $\lambda\mu^{S4}$-calculus satisfies subject reduction, strong normalization, and confluency.

As for the formulation of classical S4 modal logic in natural deduction style, the one given by Prawitz [19] is known well. However, normalization in Prawitz's system does not hold. Medeiros pointed out it, and showed normalization by giving a modified system [12]. This paper gives a stronger result than Medeiros's one, since strong normalization and confluency of $\mathrm{CND}^{\mathrm{S4}}$ is obtained from the results of the $\lambda\mu^{S4}$-calculus.

We also discuss computational interpretation of the $\lambda\mu^{S4}$-calculus. This calculus provides both mechanisms of staged computation and exception handling, because it is an extension of both the modal λ-calculus and the $\lambda\mu$-calculus. A computational interpretation of the possibility operator can be obtained via the duality of classical modal logic: $\Diamond A$ is a type of programs that can be used at *some* later stage. We consider an application of the possibility operator by giving a program example of staged computation with exception handling.

This paper is organized as follows. Section 2 introduces the classical natural deduction $\mathrm{CND}^{\mathrm{S4}}$, and shows its provability is equivalent to classical S4 modal logic. Section 3 gives the $\lambda\mu^{S4}$-calculus as the corresponding system of $\mathrm{CND}^{\mathrm{S4}}$. In Section 4, we show subject reduction, strong normalization, and confluency of the $\lambda\mu^{S4}$-calculus. Section 5 gives some discussions on the computational interpretation of $\lambda\mu^{S4}$. Finally, we conclude the paper in Section 6.

2 Classical Modal Propositional Logic

We propose a natural deduction system for classical S4 modal logic (called $\mathrm{CND}^{\mathrm{S4}}$) extending Parigot's Classical Natural Deduction (CND) [16]. CND is a natural deduction system for classical logic, and has sequents with multiple conclusions. Though Parigot gave CND for the second-order classical predicate logic, we consider the system for the $\{\supset, \neg, \Box, \Diamond\}$-fragment of classical S4 modal propositional logic for simplicity.

Definition 1 (Formulas). Formulas (denoted by $A, B, \ldots$) are defined by
$A ::= X \mid A \supset A \mid \neg A \mid \Box A \mid \Diamond A$,
where $X, Y, Z, \ldots$ are atomic formulas.

Let Γ, Δ, Σ, and Θ range over finite multisets of formulas.

The sequents of $\mathrm{CND}^{\mathrm{S4}}$ have the following form:

$$A_1, \ldots, A_n; B_1, \ldots, B_m \vdash_{\mathrm{ND}} C_1, \ldots, C_p; D_1, \ldots, D_q,$$

where $n, m, p, q \geq 0$. The parts $A_1, \ldots, A_n; B_1, \ldots, B_m$ and $C_1, \ldots, C_p; D_1, \ldots, D_q$ are the antecedent and the succedent of this sequent, respectively. Each of them is separated into two zones by the symbol ;. The classical antecedent and the classical succedent of this sequent are the parts $B_1, \ldots, B_m$ and $C_1, \ldots, C_p$, respectively. They are sometimes called the classical part of the sequent. The modal antecedent and the modal succedent of this sequent are the parts $A_1, \ldots, A_n$ and $D_1, \ldots, D_q$, respectively. They are sometimes called the modal part of the sequent. We implicitly assume $\Box$ at the head of each A_i. We also assume $\Diamond$ at the head of each D_j. The interpretation of the above sequent is given as follows: If all of $\Box A_1, \ldots \Box A_n$ and $B_1, \ldots, B_m$ are true, then some of $C_1, \ldots, C_p$ or $\Diamond D_1, \ldots, \Diamond D_q$ is true.

Definition 2 (Inference rules). Inference rules of $\mathrm{CND}^{\mathrm{S4}}$ are defined as follows.

$$\frac{}{\Sigma; \Gamma, A \vdash_{\mathrm{ND}} A, \Delta; \Theta}\,(AxC) \qquad \frac{}{\Sigma, A; \Gamma \vdash_{\mathrm{ND}} A, \Delta; \Theta}\,(AxM)$$

$$\frac{\Sigma; \Gamma, A \vdash_{\mathrm{ND}} B, \Delta; \Theta}{\Sigma; \Gamma \vdash_{\mathrm{ND}} A \supset B, \Delta; \Theta}\,(\supset I) \qquad \frac{\Sigma; \Gamma \vdash_{\mathrm{ND}} A \supset B, \Delta; \Theta \quad \Sigma; \Gamma \vdash_{\mathrm{ND}} A, \Delta; \Theta}{\Sigma; \Gamma \vdash_{\mathrm{ND}} B, \Delta; \Theta}\,(\supset E)$$

$$\frac{\Sigma; \Gamma, A \vdash_{\mathrm{ND}} \Delta; \Theta}{\Sigma; \Gamma \vdash_{\mathrm{ND}} \neg A, \Delta; \Theta}\,(\neg I) \qquad \frac{\Sigma; \Gamma \vdash_{\mathrm{ND}} \neg A, \Delta; \Theta \quad \Sigma; \Gamma \vdash_{\mathrm{ND}} A, \Delta; \Theta}{\Sigma; \Gamma \vdash_{\mathrm{ND}} \Delta; \Theta}\,(\neg E)$$

$$\frac{\Sigma; \quad \vdash_{\mathrm{ND}} A \quad ; \Theta}{\Sigma; \Gamma \vdash_{\mathrm{ND}} \Box A, \Delta; \Theta}\,(\Box I) \qquad \frac{\Sigma; \Gamma \vdash_{\mathrm{ND}} \Box A, \Delta; \Theta \quad \Sigma, A; \Gamma \vdash_{\mathrm{ND}} \Delta; \Theta}{\Sigma; \Gamma \vdash_{\mathrm{ND}} \Delta; \Theta}\,(\Box E)$$

$$\frac{\Sigma; \Gamma \vdash_{\mathrm{ND}} \Delta; A, \Theta}{\Sigma; \Gamma \vdash_{\mathrm{ND}} \Diamond A, \Delta; \Theta}\,(\Diamond I) \qquad \frac{\Sigma; \Gamma \vdash_{\mathrm{ND}} \Diamond A, \Delta; \Theta \quad \Sigma; A \vdash_{\mathrm{ND}} \quad ; \Theta}{\Sigma; \Gamma \vdash_{\mathrm{ND}} \Delta; \Theta}\,(\Diamond E) \qquad \frac{\Sigma; \Gamma \vdash_{\mathrm{ND}} A, \Delta; \Theta}{\Sigma; \Gamma \vdash_{\mathrm{ND}} \Delta; A, \Theta}\,(IR)$$

The formulas explicitly mentioned in the rules are called active. The active formula $A \supset B$ of $(\supset E)$, $\neg A$ of $(\neg E)$, $\Box A$ of $(\Box E)$, or $\Diamond A$ of $(\Diamond E)$ is said to be the major premise of each rule.

(AxC) is the axiom rule for the classical part. (AxM) is the axiom rule for the modal part. (IR) moves a formula from the classical succedent to the modal succedent. $(\supset I)$, $(\neg I)$, $(\Box I)$, and $(\Diamond I)$ are introduction rules. $(\supset E)$, $(\neg E)$, $(\Box E)$, and $(\Diamond E)$ are elimination rules.

Remark 1. (AxM) implicitly removes a $\Box$-operator, since each formula at the modal antecedent is implicitly boxed. $(\Box E)$ simply eliminates its major premise $\Box A$ without removing the $\Box$-operator. Both rules are necessary to show $\;;\; \vdash_{\mathrm{ND}} \Box A \supset A;\;$. Dually, (IR) implicitly introduces a $\Diamond$-operator, since each formula at the modal succedent is implicitly diamonded. $(\Diamond I)$ specifies the $\Diamond$-operator that is implicitly introduced by (IR). We need both rules to show $\;;\; \vdash_{\mathrm{ND}} A \supset \Diamond A;\;$.

Remark 2. The separation symbol ; in the sequents of $\mathrm{CND}^{\mathrm{S4}}$ is necessary for normalization. Let $\mathrm{CND}^{\mathrm{S4}\prime}$ be a natural deduction system obtained by replacing sequents

$\Sigma;\Gamma \vdash_{\mathrm{ND}} \Delta;\Theta$ of CND$^{\mathrm{S4}}$ by $\Box\Sigma,\Gamma \vdash \Delta,\Diamond\Theta$. Then normalization of CND$^{\mathrm{S4}\prime}$ does not hold. For example, the following proof is not normalizable.

$$\dfrac{\dfrac{\dfrac{\dfrac{}{\Box B \vdash \Box B}}{\Box B \vdash \Box\Box B}\,(\Box I)}{\vdash \Box B \supset \Box\Box B}\,(\supset I) \quad \dfrac{\dfrac{}{\Box A \supset \Box B \vdash \Box A \supset \Box B} \quad \dfrac{}{\Box A \vdash \Box A}}{\Box A \supset \Box B, \Box A \vdash \Box B}\,(\supset E)}{\Box A \supset \Box B, \Box A \vdash \Box\Box B}\,(\supset E)$$

CND$^{\mathrm{S4}}$ admits weakening and contraction rules.

Lemma 1 (Weakening). *Assume $\Sigma \subseteq \Sigma'$, $\Gamma \subseteq \Gamma'$, $\Delta \subseteq \Delta'$, and $\Theta \subseteq \Theta'$. Then $\Sigma;\Gamma \vdash_{\mathrm{ND}} \Delta;\Theta$ implies $\Sigma';\Gamma' \vdash_{\mathrm{ND}} \Delta';\Theta'$.*

Proof. The claim is shown by induction on the length of proofs.

In the following, if $\Sigma;\Gamma \vdash_{\mathrm{ND}} \Delta;\Theta$ is obtained from $\Sigma';\Gamma' \vdash_{\mathrm{ND}} \Delta';\Theta'$ by applying rules $R_1,\ldots,R_n$ several times, then we write $\dfrac{\Sigma';\Gamma' \vdash_{\mathrm{ND}} \Delta';\Theta'}{\Sigma;\Gamma \vdash_{\mathrm{ND}} \Delta;\Theta}\, R_1,\ldots,R_n$. We will write R' if an elimination rule R is used with weakening. For example, $(\neg E)'$ is used as follows:

$$\dfrac{\Sigma_1,\Sigma_2;\Gamma_1,\Gamma_2 \vdash_{\mathrm{ND}} \neg A,\Delta_1,\Delta_2;\Theta_1,\Theta_2 \quad \Sigma_2,\Sigma_3;\Gamma_2,\Gamma_3 \vdash_{\mathrm{ND}} A,\Delta_2,\Delta_3;\Theta_2,\Theta_3}{\Sigma_1,\Sigma_2,\Sigma_3;\Gamma_1,\Gamma_2,\Gamma_3 \vdash_{\mathrm{ND}} \Delta_1,\Delta_2,\Delta_3;\Theta_1,\Theta_2,\Theta_3}\,(\neg E)' .$$

Lemma 2. (1) *$\Sigma,A;\Gamma \vdash_{\mathrm{ND}} \Delta;\Theta$ holds if and only if $\Sigma;\Box A,\Gamma \vdash_{\mathrm{ND}} \Delta;\Theta$ holds.*
(2) *$\Sigma;\Gamma \vdash_{\mathrm{ND}} \Delta;A,\Theta$ holds if and only if $\Sigma;\Gamma \vdash_{\mathrm{ND}} \Delta,\Diamond A;\Theta$ holds.*

Proof. (1) Assume $\Sigma,A;\Gamma \vdash_{\mathrm{ND}} \Delta;\Theta$. Then we have $\Sigma;\Box A,\Gamma \vdash_{\mathrm{ND}} \Delta;\Theta$ by (AxC), and $(\Box E)'$. Conversely, assume $\Sigma;\Box A,\Gamma \vdash_{\mathrm{ND}} \Delta;\Theta$, then we have $\Sigma;\Gamma \vdash_{\mathrm{ND}} \neg\Box A,\Delta;\Theta$ by $(\neg I)$. Here $\Sigma,A;\Gamma \vdash_{\mathrm{ND}} \Box A,\Delta;\Theta$ is shown by (AxM) and $(\Box I)$. Therefore we obtain $\Sigma,A;\Gamma \vdash_{\mathrm{ND}} \Delta;\Theta$ by $(\neg E)'$.

(2) We obtain $\Sigma;\Gamma \vdash_{\mathrm{ND}} \Delta,\Diamond A;\Theta$ from $\Sigma;\Gamma \vdash_{\mathrm{ND}} \Delta;A,\Theta$ by using (IR). Conversely, assume $\Sigma;\Gamma \vdash_{\mathrm{ND}} \Delta,\Diamond A;\Theta$. Then we have $\Sigma;\Gamma \vdash_{\mathrm{ND}} \Delta;A,\Theta$ by (AxC), (IR), and $(\Diamond E)'$.

Lemma 3 (Left contraction). (1) *If $\Sigma,A,A;\Gamma \vdash_{\mathrm{ND}} \Delta;\Theta$, then $\Sigma,A;\Gamma \vdash_{\mathrm{ND}} \Delta;\Theta$ hold.*
(2) *If $\Sigma;\Gamma,A,A \vdash_{\mathrm{ND}} \Delta;\Theta$, then $\Sigma;\Gamma,A \vdash_{\mathrm{ND}} \Delta;\Theta$ hold.*

Proof. (1) and (2) are shown by induction on the length of proofs.

Lemma 4 (Right contraction). (1) *If $\Sigma;\Gamma \vdash_{\mathrm{ND}} \Delta,A,A;\Theta$, then $\Sigma;\Gamma \vdash_{\mathrm{ND}} \Delta,A;\Theta$ hold.*
(2) *If $\Sigma;\Gamma \vdash_{\mathrm{ND}} \Delta;\Theta,A,A$, then $\Sigma;\Gamma \vdash_{\mathrm{ND}} \Delta;\Theta,A$ hold.*

Proof. (1) Suppose we have $\Sigma;\Gamma \vdash_{\mathrm{ND}} \Delta,A,A;\Theta$. Then $\Sigma;\Gamma \vdash_{\mathrm{ND}} \Delta,A;\Theta$ is obtained by:

$$\dfrac{\begin{array}{c}\vdots\\ ;\vdash_{\mathrm{ND}} \neg\neg A \supset A;\end{array} \quad \dfrac{\dfrac{\dfrac{\dfrac{}{;\neg A \vdash_{\mathrm{ND}} \neg A;} \quad \Sigma;\Gamma \vdash_{\mathrm{ND}} \Delta,A,A;\Theta}{\Sigma;\Gamma,\neg A \vdash_{\mathrm{ND}} \Delta,A;\Theta}\,(\neg E)' \quad \dfrac{}{;\neg A \vdash_{\mathrm{ND}} \neg A;}}{\Sigma;\Gamma,\neg A \vdash_{\mathrm{ND}} \Delta;\Theta}\,(\neg E)'}{\Sigma;\Gamma \vdash_{\mathrm{ND}} \neg\neg A,\Delta;\Theta}\,(\neg I)}{\Sigma;\Gamma \vdash_{\mathrm{ND}} \Delta,A;\Theta}\,(\supset E)'$$

We claim that $;\vdash_{\mathrm{ND}} \neg\neg A \supset A;$ is proved from (AxC) by using $(\neg I)$ and $(\neg E)$. (2) is shown by using (1) and Lemma 2.

We sometimes write weakening and left contraction rules as follows.

$$\frac{\Sigma;\Gamma \vdash_{\mathrm{ND}} \Delta;\Theta}{\Sigma';\Gamma' \vdash_{\mathrm{ND}} \Delta';\Theta'}\ (\mathit{Wk}) \quad \frac{\Sigma;\Gamma,A,A \vdash_{\mathrm{ND}} \Delta;\Theta}{\Sigma;\Gamma,A \vdash_{\mathrm{ND}} \Delta;\Theta}\ (\mathit{CtrC}_L) \quad \frac{\Sigma,A,A;\Gamma \vdash_{\mathrm{ND}} \Delta;\Theta}{\Sigma,A;\Gamma \vdash_{\mathrm{ND}} \Delta;\Theta}\ (\mathit{CtrM}_L)$$

$$\frac{\Sigma;\Gamma \vdash_{\mathrm{ND}} A,A,\Delta;\Theta}{\Sigma;\Gamma \vdash_{\mathrm{ND}} A,\Delta;\Theta}\ (\mathit{CtrC}_R) \quad \frac{\Sigma;\Gamma \vdash_{\mathrm{ND}} \Delta;A,A,\Theta}{\Sigma;\Gamma \vdash_{\mathrm{ND}} \Delta;A,\Theta}\ (\mathit{CtrM}_R)$$

where $\Sigma \subseteq \Sigma'$, $\Gamma \subseteq \Gamma'$, $\Delta \subseteq \Delta'$, and $\Theta \subseteq \Theta'$. (CtrC_R) and (CtrC_L) are the right and left contraction rules for the classical part. (CtrM_R) and (CtrM_L) are the right and left contraction rules for the modal part.

$\mathrm{CND}^{\mathrm{S4}}$ is equivalent to the classical S4 modal logic in the sense of provability. Here we remember the sequent calculus style formulation of classical S4 modal logic.

The sequent calculus have sequents of the form $\Gamma \vdash_{\mathrm{SC}} \Delta$. The antecedent and succedent of a sequent $\Gamma \vdash_{\mathrm{SC}} \Delta$ is defined by Γ and Δ, respectively. The interpretation of a sequent $\Gamma \vdash_{\mathrm{SC}} \Delta$ is given as follows: If all formulas in Γ are true, then some formula in Δ is true.

The inference rules of the sequent calculus for classical modal logic are given as usual. We display only the rules for modal operators:

$$\frac{\Box\Gamma \vdash_{\mathrm{SC}} A,\Diamond\Delta}{\Box\Gamma \vdash_{\mathrm{SC}} \Box A,\Diamond\Delta}\ (\Box R) \quad \frac{\Gamma,A \vdash_{\mathrm{SC}} \Delta}{\Gamma,\Box A \vdash_{\mathrm{SC}} \Delta}\ (\Box L) \quad \frac{\Gamma \vdash_{\mathrm{SC}} A,\Delta}{\Gamma \vdash_{\mathrm{SC}} \Diamond A,\Delta}\ (\Diamond R) \quad \frac{\Box\Gamma,A \vdash_{\mathrm{SC}} \Diamond\Delta}{\Box\Gamma,\Diamond A \vdash_{\mathrm{SC}} \Diamond\Delta}\ (\Diamond L)$$

where $\Box\Gamma$ is $\Box A_1,\ldots,\Box A_n$ if Γ is $A_1,\ldots,A_n$, $\Diamond\Delta$ is $\Diamond B_1,\ldots,\Diamond B_m$ if Δ is $B_1,\ldots,B_m$.

Theorem 1. *$\Sigma;\Gamma \vdash_{\mathrm{ND}} \Delta;\Theta$ is provable if and only if $\Box\Sigma,\Gamma \vdash_{\mathrm{SC}} \Delta,\Diamond\Theta$ is provable.*

Proof. The *only-if*-part is shown by induction on the proof of $\mathrm{CND}^{\mathrm{S4}}$. To show the *if*-part, we first show the following claim (we call this claim (*)): $\Box\Sigma,\Gamma \vdash_{\mathrm{SC}} \Delta,\Diamond\Theta$ implies $;\Box\Sigma,\Gamma \vdash_{\mathrm{ND}} \Delta,\Diamond\Theta;$. If we have this claim, we can show $\Sigma;\Gamma \vdash_{\mathrm{ND}} \Delta;\Theta$ by using Lemma 2. The claim (*) is shown by induction on the proof of the sequent calculus. We consider the cases $(\Box R)$, $(\Box L)$, $(\Diamond R)$, and $(\Diamond L)$.

The case of $(\Box R)$: Assume that $\Box\Sigma,\Gamma \vdash_{\mathrm{SC}} \Box A,\Delta,\Diamond\Theta$ is proved from $\Box\Sigma,\Gamma \vdash_{\mathrm{SC}} A,\Delta,\Diamond\Theta$. By the condition of $(\Box R)$, Γ and Δ should be $\Box\Gamma'$ and $\Diamond\Delta'$ for some Γ' and Δ', respectively. By the induction hypothesis, we have $;\Box\Sigma,\Box\Gamma' \vdash_{\mathrm{ND}} A,\Diamond\Delta',\Diamond\Theta;$. Then we have $\Sigma,\Gamma'; \ \vdash_{\mathrm{ND}} A;\Delta',\Theta$ by Lemma 2. Thus $\Sigma,\Gamma'; \ \vdash_{\mathrm{ND}} \Box A;\Delta',\Theta$ is shown by $(\Box I)$. By using Lemma 2 again, we obtain $;\Box\Sigma,\Box\Gamma' \vdash_{\mathrm{ND}} \Box A,\Diamond\Delta',\Diamond\Theta;$. The case of $(\Diamond L)$ is shown similarly. The case of $(\Box L)$ is proved by using the induction hypothesis and Lemma 2. The case of $(\Diamond R)$ is proved by using the induction hypothesis.

We define the normalization procedure of $\mathrm{CND}^{\mathrm{S4}}$. Each reduction step of the procedure removes a formula occurrence (we call cut-formula) that is the consequence of an introduction rule and the major premise of an elimination rule. We distinguish a reduction step between *logical* reduction and *structural* reduction according to the occurrence of its cut-formula. A reduction step is called logical when its cut-formula is the major premise of an elimination rule, and it is introduced by the immediate preceding rule. A reduction step is called structural when its cut-formula is the major premise of an elimination rule, and it is not active in the immediate preceding rule.

The logical reduction and the structural reduction are defined as well as that of CND [16] when its cut-formula is $A \supset B$ or $\neg A$. We give the definition when the cut-formula is $\Box A$ or $\Diamond A$.

Logical $\Box$-reduction: We assume that the cut-formula has the form $\Box A$, and is the major premise of a $(\Box E)$ rule, and is introduced by the immediate preceding $(\Box I)$ rule. Then we have the following proof:

$$\dfrac{\dfrac{\begin{array}{c}\vdots\\ \Sigma;\quad \vdash_{\mathrm{ND}} A;\Theta\end{array}}{\Sigma;\Gamma\vdash_{\mathrm{ND}}\Box A,\Delta;\Theta}\,(\Box I) \qquad \begin{array}{c}\overline{\Sigma_0,A;\Gamma_0\vdash_{\mathrm{ND}}A,\Delta_0;\Theta_0}\\ \vdots\\ \Sigma,A;\Gamma\vdash_{\mathrm{ND}}\Delta;\Theta\end{array}}{\Sigma;\Gamma\vdash_{\mathrm{ND}}\Delta;\Theta}\,(\Box E)\ .$$

This proof is reduced to the proof:

$$\begin{array}{c}\dfrac{\begin{array}{c}\vdots\\ \Sigma;\quad \vdash_{\mathrm{ND}} A;\Theta\end{array}}{\Sigma,\Sigma_0;\Gamma_0\vdash_{\mathrm{ND}}A,\Delta_0;\Theta,\Theta_0}\,(Wk)\\ \vdots\\ \begin{array}{c}\Sigma,\Sigma;\Gamma\vdash_{\mathrm{ND}}\Delta;\Theta,\Theta\\ \hline\hline \Sigma;\Gamma\vdash_{\mathrm{ND}}\Delta;\Theta\end{array}\ (CtrM_L),(CtrM_R)\end{array}\ .$$

Logical $\Diamond$-reduction: We assume that the cut-formula has the form $\Diamond A$, and is the major premise of a $(\Diamond E)$ rule, and is introduced by the immediate preceding $(\Diamond I)$ rule. Then we have the following proof:

$$\dfrac{\dfrac{\begin{array}{c}\dfrac{\begin{array}{c}\vdots\\ \Sigma_0;\Gamma_0\vdash_{\mathrm{ND}}A,\Delta_0;\Theta_0\end{array}}{\Sigma_0;\Gamma_0\vdash_{\mathrm{ND}}\Delta_0;A,\Theta_0}\,(IR)\\ \vdots\\ \Sigma;\Gamma\vdash_{\mathrm{ND}}\Delta;A,\Theta\end{array}}{\Sigma;\Gamma\vdash_{\mathrm{ND}}\Diamond A,\Delta;\Theta}\,(\Diamond I) \qquad \begin{array}{c}\vdots\\ \Sigma;A\vdash_{\mathrm{ND}}\quad;\Theta\end{array}}{\Sigma;\Gamma\vdash_{\mathrm{ND}}\Delta;\Theta}\,(\Diamond E)\ .$$

This proof is reduced to the proof:

$$\begin{array}{c}\dfrac{\dfrac{\begin{array}{c}\vdots\\ \Sigma;A\vdash_{\mathrm{ND}}\quad;\Theta\end{array}}{\Sigma;\quad\vdash_{\mathrm{ND}}\neg A;\Theta}\,(\neg I) \qquad \begin{array}{c}\vdots\\ \Sigma_0;\Gamma_0\vdash_{\mathrm{ND}}A,\Delta_0;\Theta_0\end{array}}{\Sigma,\Sigma_0;\Gamma_0\vdash_{\mathrm{ND}}\Delta_0;\Theta,\Theta_0}\,(\neg E)'\\ \vdots\\ \begin{array}{c}\Sigma,\Sigma;\Gamma\vdash_{\mathrm{ND}}\Delta;\Theta,\Theta\\ \hline\hline \Sigma;\Gamma\vdash_{\mathrm{ND}}\Delta;\Theta\end{array}\ (CtrM_L),(CtrM_R)\end{array}\ .$$

Structural $\Box$-reduction: We assume that the cut-formula has the form $\Box A$, and is the major premise of a $(\Box E)$ rule, and is not active in the immediate preceding rule. Then we have the following proof:

$$\dfrac{\begin{array}{c}\vdots\ \pi_0\\ \Sigma_0;\Gamma_0\vdash_{\mathrm{ND}}\Box A,\Delta_0;\Theta_0\\ \vdots\ \pi_1\\ \Sigma;\Gamma\vdash_{\mathrm{ND}}\Box A,\Delta;\Theta\end{array} \qquad \begin{array}{c}\vdots\\ \Sigma,A;\Gamma\vdash_{\mathrm{ND}}\Delta;\Theta\end{array}}{\Sigma;\Gamma\vdash_{\mathrm{ND}}\Delta;\Theta}\,(\Box E)\ ,$$

where $\Box A$ is active in the last rule of π_0, and not active in all rules of π_1. This proof is reduced to the following proof:

$$\begin{array}{c}\dfrac{\begin{array}{c}\vdots\ \pi_0\\ \Sigma_0;\Gamma_0\vdash_{\mathrm{ND}}\Box A,\Delta_0;\Theta_0\end{array} \qquad \begin{array}{c}\vdots\\ \Sigma,A;\Gamma\vdash_{\mathrm{ND}}\Delta;\Theta\end{array}}{\Sigma,\Sigma_0;\Gamma,\Gamma_0\vdash_{\mathrm{ND}}\Delta,\Delta_0;\Theta,\Theta_0}\,(\Box E)'\\ \vdots\ \pi_1\\ \begin{array}{c}\Sigma,\Sigma;\Gamma,\Gamma\vdash_{\mathrm{ND}}\Delta,\Delta;\Theta,\Theta\\ \hline\hline \Sigma;\Gamma\vdash_{\mathrm{ND}}\Delta;\Theta\end{array}\ (CtrC_L),(CtrC_R),(CtrM_L),(CtrM_R)\end{array}\ .$$

Structural $\Diamond$-reduction: We assume that the cut-formula has the form $\Diamond A$, and is the major premise of a $(\Diamond E)$ rule, and is not active in the immediate preceding rule. Then we have the following proof:

$$\dfrac{\begin{array}{c}\vdots\ \pi_0\\ \Sigma_0;\Gamma_0 \vdash_{\mathrm{ND}} \Diamond A,\Delta_0;\Theta_0\\ \vdots\ \pi_1\\ \Sigma;\Gamma \vdash_{\mathrm{ND}} \Diamond A,\Delta;\Theta\end{array} \qquad \begin{array}{c}\vdots\\ \Sigma;A \vdash_{\mathrm{ND}}\ ;\Theta\end{array}}{\Sigma;\Gamma \vdash_{\mathrm{ND}} \Delta;\Theta}\,(\Diamond E)\ ,$$

where $\Diamond A$ is active in the last rule of π_0, and not active in all rules of π_1. This is reduced to the following proof:

$$\begin{array}{c}\dfrac{\begin{array}{c}\vdots\\ \Sigma_0;\Gamma_0 \vdash_{\mathrm{ND}} \Diamond A,\Delta_0;\Theta_0\end{array} \quad \begin{array}{c}\vdots\\ \Sigma;\ A \vdash_{\mathrm{ND}}\ ;\Theta\end{array}}{\Sigma_0,\Sigma;\Gamma_0,\Gamma \vdash_{\mathrm{ND}} \Delta_0,\Delta;\Theta_0,\Theta}\,(\Diamond E)'\\ \vdots\\ \begin{array}{c}\Sigma,\Sigma;\Gamma,\Gamma \vdash_{\mathrm{ND}} \Delta,\Delta;\Theta,\Theta\\ \hline\hline \Sigma;\Gamma \vdash_{\mathrm{ND}} \Delta;\Theta\end{array}\ (CtrC_L),(CtrC_R),(CtrM_L),(CtrM_R)\end{array}\ .$$

In the next section, we introduce the $\lambda\mu^{S4}$-calculus that corresponds to CND$^{\mathrm{S4}}$. The reduction procedure of CND$^{\mathrm{S4}}$ satisfies confluency and normalizability. They are obtained from confluency (Theorem 4) and strong normalizability (Theorem 2) of the $\lambda\mu^{S4}$-calculus.

3 Modal $\lambda\mu$-calculus

This section gives the definition of the modal $\lambda\mu$-calculus (called $\lambda\mu^{S4}$).

Definition 3 (Types). Let $X,Y,Z\ldots$ range over type variables. A type of the $\lambda\mu^{S4}$-calculus (denoted by $T,U,\ldots$) is either the special type $\bot$ or a normal type (denoted by $A,B,\ldots$) defined as follows:

Types $T ::= A \mid \bot$
Normal types $A ::= X \mid A \supset A \mid \neg A \mid \Box A \mid \Diamond A$

There are four kinds of variables for the $\lambda\mu^{S4}$-calculus, called classical variables, classical covariables, modal variables, and modal covariables. They are respectively corresponding to classical antecedent, classical succedent, modal antecedent, and modal succedent of sequents in CND$^{\mathrm{S4}}$.

Then we define expressions of the $\lambda\mu^{S4}$-calculus.

Definition 4 (Expressions). Let $x,y,\ldots$ range over classical variables, $a,b,\ldots$ range over classical covariables, $\chi,\upsilon,\ldots$ range over modal variables, and $\alpha,\beta,\ldots$ range over modal covariables. An expression (denoted by $E,F,\ldots$) of the $\lambda\mu^{S4}$-calculus is either a term (denoted by $M,N,\ldots$) or a statement (denoted by $R,S,\ldots$). They are defined as follows.

Expressions $E ::= M \mid R$
Terms $M ::= x \mid \lambda x.M \mid MM \mid \lambda x.R \mid \mu a.R \mid \chi \mid \Box M \mid \Diamond\alpha.R$
Statements $R ::= [a]M \mid M \cdot M \mid \mathtt{let}\ \Box\chi\ \mathtt{be}\ M\ \mathtt{in}\ R \mid \alpha M \mid \mathtt{dia}\langle x.R\rangle(M)$

$\lambda x.M$ binds the classical variable x in M. $\lambda x.R$ and $\mathtt{dia}\langle x.R\rangle(M)$ bind the classical variable x in R. $\mu a.R$ binds the classical covariable a in R. $\mathtt{let}\ \Box\chi\ \mathtt{be}\ M\ \mathtt{in}\ R$ binds the modal variable χ in R. $\Diamond\alpha.R$ binds the modal covariable α in R. A variable is called free in an expression if it is not bound in the expression. The set of free variables in an expression E is denoted by $FV(E)$.

Terms are extensions of unnamed terms of Parigot's $\lambda\mu$-calculus [16]. They are expressions for normal types in the type system. Statements are extensions of named terms of the $\lambda\mu$-calculus. They are expressions for $\perp$ type.

Substitution $E[{}^{M}/_{x}]$ for a classical variable x is defined by the expression obtained from E replacing each free occurrence of x by M. Substitution $E[{}^{M}/_{\chi}]$ for a modal variable χ is defined by the expression obtained from E replacing each free occurrence of χ by M. Substitution $E[{}^{M}/_{\alpha}]$ for a modal covariable α is defined by the result recursively replacing each subexpression of the form αN in E by $M \cdot (N[{}^{M}/_{\alpha}])$. We write $E[{}^{\beta}/_{\alpha}]$ for the expression obtained from E replacing α by β.

An expression with one hole $\{-\}$ that accepts a term is called contexts. $C\{M\}$ is the expression obtained from a context C by putting a term M in the hole. Elimination contexts (denoted by $\mathcal{E}$) are contexts defined as follows:

$\mathcal{E} ::= [a]\{-\} \mid \{-\} \cdot N \mid [a]\{-\}N \mid \texttt{let}\ \Box\chi\ \texttt{be}\ \{-\}\ \texttt{in}\ S \mid \texttt{dia}\langle x.S\rangle(\{-\})$.

Then we define substitution $E[{}^{\mathcal{E}}/_{[a]\{-\}}]$ for a classical covariable a by the result recursively replacing each subexpression of the form $[a]N$ in E by $\mathcal{E}\{\, M[{}^{\mathcal{E}}/_{[a]\{-\}}]\,\}$. We sometimes write $E[{}^{b}/_{a}]$ for $E[{}^{[b]\{-\}}/_{[a]\{-\}}]$.

Definition 5 (Typing judgments and typing rules). A modal typing context (denoted by Σ) is a set $\chi_1 : A_1, \ldots, \chi_n : A_n$ of modal variable declarations. A classical typing context (denoted by Γ) is a set $x_1 : B_1, \ldots, x_m : B_m$ of classical variable declarations. A classical typing cocontext (denoted by Δ) is a set $a_1 : C_1, \ldots, a_p : C_p$ of classical covariable declarations. A modal typing cocontext (denoted by Θ) is a set $\alpha_1 : D_1, \ldots, \alpha_q : D_q$ of modal covariable declarations. We assume that any two variables in a typing context and cocontext are distinct.

A typing judgment (denoted by J) for the $\lambda\mu^{S4}$-calculus takes either the form $\Sigma; \Gamma \vdash M\colon A \mid \Delta; \Theta$, or the form $\Sigma; \Gamma \vdash S : \perp \mid \Delta; \Theta$, We will write $\Sigma; \Gamma \vdash E\colon T \mid \Delta; \Theta$ for denoting the two forms of typing judgments together.

The intuitive meaning of a typing judgment J is given by the sequent J^- of $\mathrm{CND}^{\mathrm{S4}}$ defined as follows. If J is $\Sigma; \Gamma \vdash M\colon A \mid \Delta; \Theta$, then J^- is defined by $\Sigma^-; \Gamma^- \vdash_{\mathrm{ND}} A, \Delta^-; \Theta^-$, where Σ^- is $A_1, \ldots, A_n$ if Σ is $\chi_1\colon A_1, \ldots, \chi_n\colon A_n$, Γ^- is $B_1, \ldots, B_m$ if Γ is $x_1\colon B_1, \ldots, x_m\colon B_m$, Δ^- is $C_1, \ldots, C_p$ if Δ is $a_1\colon C_1, \ldots, a_p\colon C_p$, and Θ^- is $D_1, \ldots, D_q$ if Θ is $\alpha_1\colon D_1, \ldots, \alpha_q\colon D_q$. We also give J^- by $\Sigma^-; \Gamma^- \vdash_{\mathrm{ND}} \Delta^-; \Theta^-$ if J is $\Sigma; \Gamma \vdash S : \perp \mid \Delta; \Theta$. The computational interpretation of typing judgments of the $\lambda\mu^{S4}$-calculus will be discussed in Section 5.

Definition 6 (Typing rules). The typing rules for the $\lambda\mu^{S4}$-calculus are defined as follows.

$$\frac{}{\Sigma; \Gamma, x\colon A \vdash x\colon A \mid \Delta; \Theta}\ (AxC) \qquad \frac{}{\Sigma, \chi\colon A; \Gamma \vdash \chi\colon A \mid \Delta; \Theta}\ (AxM)$$

$$\frac{\Sigma; \Gamma, x\colon A \vdash M\colon B \mid \Delta; \Theta}{\Sigma; \Gamma \vdash \lambda x.M\colon A \supset B \mid \Delta; \Theta}\ (\supset I) \qquad \frac{\Sigma; \Gamma \vdash M\colon A \supset B \mid \Delta; \Theta \quad \Sigma; \Gamma \vdash N\colon A \mid \Delta; \Theta}{\Sigma; \Gamma \vdash MN\colon B \mid \Delta; \Theta}\ (\supset E)$$

$$\frac{\Sigma; \Gamma, x\colon A \vdash S : \perp \mid \Delta; \Theta}{\Sigma; \Gamma \vdash \lambda x.S : \neg A \mid \Delta; \Theta}\ (\neg I) \qquad \frac{\Sigma; \Gamma \vdash M\colon \neg A \mid \Delta; \Theta \quad \Sigma; \Gamma \vdash N\colon A \mid \Delta; \Theta}{\Sigma; \Gamma \vdash M \cdot N\colon \perp \mid \Delta; \Theta}\ (\neg E)$$

$$\frac{\Sigma;\ \ \vdash M\colon A \mid\ \ ;\Theta}{\Sigma;\Gamma \vdash \Box M\colon \Box A \mid \Delta;\Theta}\,(\Box I) \qquad \frac{\Sigma;\Gamma \vdash M\colon \Box A \mid \Delta;\Theta \quad \Sigma,\chi\colon A;\Gamma \vdash S : \bot \mid \Delta;\Theta}{\Sigma;\Gamma \vdash \mathtt{let}\ \Box\chi\ \mathtt{be}\ M\ \mathtt{in}\ S : \bot \mid \Delta;\Theta}\,(\Box E)$$

$$\frac{\Sigma;\Gamma \vdash S : \bot \mid \Delta;\Theta,\alpha\colon A}{\Sigma;\Gamma \vdash \Diamond\alpha.S : \Diamond A \mid \Delta;\Theta}\,(\Diamond I) \qquad \frac{\Sigma;\Gamma \vdash M\colon \Diamond A \mid \Delta;\Theta \quad \Sigma;x\colon A \vdash S : \bot \mid\ \ ;\Theta}{\Sigma;\Gamma \vdash \mathtt{dia}\langle x.S\rangle(M)\colon \bot \mid \Delta;\Theta}\,(\Diamond E)$$

$$\frac{\Sigma;\Gamma \vdash M\colon A \mid \Delta;\Theta}{\Sigma;\Gamma \vdash \alpha M\colon \bot \mid \Delta;\Theta,\alpha\colon A}\,(Pass_M)$$

$$\frac{\Sigma;\Gamma \vdash M\colon A \mid \Delta;\Theta}{\Sigma;\Gamma \vdash [a]M\colon \bot \mid \Delta,a\colon A;\Theta}\,(Pass_C) \qquad \frac{\Sigma;\Gamma \vdash S : \bot \mid \Delta,a\colon A;\Theta}{\Sigma;\Gamma \vdash \mu a.S : A \mid \Delta;\Theta}\,(Act)$$

Note that J is derivable in $\lambda\mu^{S4}$ implies J^- is provable in $\mathrm{CND}^{\mathrm{S4}}$. On the other hand, we can extract expressions of $\lambda\mu^{S4}$ from proofs of $\mathrm{CND}^{\mathrm{S4}}$. Each inference rule (*AxC*), (*AxM*), ($\supset I$), ($\supset E$), ($\neg I$), ($\neg E$), ($\Box I$), ($\Box E$), ($\Diamond I$), or ($\Diamond E$) of $\mathrm{CND}^{\mathrm{S4}}$ is interpreted as the typing rule of $\lambda\mu^{S4}$ with the same name. ($Pass_M$) simulates (*IR*).

Definition 7 (Reduction). The one-step reduction relation $\longrightarrow$ is defined as the compatible closure of the following reduction rules.

($\beta\supset$) $(\lambda x.M)N \longrightarrow M[{}^{N}/_{x}]$
($\beta\neg$) $(\lambda x.S)\cdot N \longrightarrow S[{}^{N}/_{x}]$
($\beta\Box$) $\mathtt{let}\ \Box\chi\ \mathtt{be}\ \Box M\ \mathtt{in}\ S \longrightarrow S[{}^{M}/_{\chi}]$
($\beta\Diamond$) $\mathtt{dia}\langle x.S\rangle(\Diamond\alpha.R) \longrightarrow R[{}^{\lambda x.S}/_{\alpha}]$
($\mu\supset$) $(\mu a.S)N \longrightarrow \mu b.S[{}^{[b]\{-\}N}/_{[a]\{-\}}] \qquad (b \notin FV(S) \cup FV(N))$
($\mu\neg$) $(\mu a.S)\cdot N \longrightarrow S[{}^{\{-\}\cdot N}/_{[a]\{-\}}]$
($\mu\Box$) $\mathtt{let}\ \Box\chi\ \mathtt{be}\ (\mu a.S)\ \mathtt{in}\ R \longrightarrow S[{}^{\mathtt{let}\ \Box\chi\ \mathtt{be}\ \{-\}\ \mathtt{in}\ R}/_{[a]\{-\}}]$
($\mu\Diamond$) $\mathtt{dia}\langle x.R\rangle(\mu a.S) \longrightarrow S[{}^{\mathtt{dia}\langle x.R\rangle(-)}/_{[a]\{-\}}]$
(*rn*) $[b]\mu a.S \longrightarrow S[{}^{[b]\{-\}}/_{[a]\{-\}}]$
($\eta\mu$) $\mu a.[a]M \longrightarrow M \qquad (a \notin FV(M))$

We write $\longrightarrow^+$ and $\longrightarrow^*$ for the transitive closure and the reflexive transitive closure of $\longrightarrow$, respectively.

An expression E is called normal if there is no expression F such that $E \longrightarrow F$.

We claim that each reduction rule ($\beta\supset$), ($\beta\neg$), ($\beta\Box$), or ($\beta\Diamond$) is interpreted as the logical reduction of each connectives in $\mathrm{CND}^{\mathrm{S4}}$. Each reduction rule ($\mu\supset$), ($\mu\neg$), ($\mu\Box$), or ($\mu\Diamond$) is interpreted as the structural reduction of each connectives in $\mathrm{CND}^{\mathrm{S4}}$. (*rn*) and ($\eta\mu$) are interpreted as identity in $\mathrm{CND}^{\mathrm{S4}}$.

4 Subject Reduction, Confluence, and Strong Normalization of the $\lambda\mu^{S4}$-Calculus

In this section, we show subject reduction, strong normalization, and confluence of the $\lambda\mu^{S4}$-calculus.

Lemma 5 (Weakening of $\lambda\mu^{S4}$). *Let $\Sigma \subseteq \Sigma'$, $\Gamma \subseteq \Gamma'$, $\Delta \subseteq \Delta'$, and $\Theta \subseteq \Theta'$. Then if $\Sigma;\Gamma \vdash E\colon T \mid \Delta;\Theta$ is derivable, then $\Sigma';\Gamma' \vdash E\colon T \mid \Delta';\Theta'$ holds.*

Proof. This claim is shown by induction on the structure of E.

Lemma 6 (Substitution). (1) *If* $\Sigma; \Gamma, x\colon A \vdash E\colon T \,|\, \Delta; \Theta$ *and* $\Sigma; \Gamma \vdash M\colon A \,|\, \Delta; \Theta$ *are derivable, then* $\Sigma; \Gamma \vdash E[{}^{M}/_{x}]\colon T \,|\, \Delta; \Theta$ *holds.*

(2) *If* $\Sigma, \chi\colon A; \Gamma \vdash E\colon T \,|\, \Delta; \Theta$ *and* $\Sigma; \ \vdash M\colon A \,|\ \ ; \Theta$ *are derivable, then* $\Sigma; \Gamma \vdash E[{}^{M}/_{\chi}]\colon T \,|\, \Delta; \Theta$ *holds.*

(3) *If* $\Sigma; \Gamma \vdash E\colon T \,|\, \Delta, a\colon A; \Theta$ *and* $\Sigma; \Gamma, x\colon A \vdash \mathcal{E}\{x\}\colon \bot \,|\, \Delta; \Theta$ *are derivable, then* $\Sigma; \Gamma \vdash E[{}^{\mathcal{E}}/_{[a]\{-\}}]\colon T \,|\, \Delta; \Theta$ *holds.*

(4) *If* $\Sigma; \Gamma \vdash E\colon T \,|\, \Delta; \Theta, \alpha\colon A$ *and* $\Sigma; \Gamma, x\colon A \vdash S : \bot \,|\, \Delta; \Theta$ *are derivable, then* $\Sigma; \Gamma \vdash E[{}^{\lambda x.S}/_{\alpha}]\colon T \,|\, \Delta; \Theta$ *holds.*

(5) *If* $\Sigma; \Gamma \vdash E\colon T \,|\, \Delta; \Theta, \alpha\colon A$ *and* $\Sigma; \Gamma, x\colon A \vdash \beta x\colon \bot \,|\, \Delta; \Theta$ *are derivable, then* $\Sigma; \Gamma \vdash E[{}^{\beta}/_{\alpha}]\colon T \,|\, \Delta; \Theta$ *holds.*

Proof. They are shown by induction on the structure of E.

By using substitution lemma, we can show that contraction rules of $\mathrm{CND}^{\mathrm{S4}}$ are admissible in $\lambda\mu^{S4}$.

Lemma 7 (Contraction of $\lambda\mu^{S4}$). (1) *If* $\Sigma; \Gamma, x\colon A, y\colon A \vdash E\colon T \mid \Delta; \Theta$ *is derivable, then* $\Sigma; \Gamma, y\colon A \vdash E[{}^{y}/_{x}]\colon T \mid \Delta; \Theta$ *holds.*

(2) *If* $\Sigma, \chi\colon A, \upsilon\colon A; \Gamma \vdash E\colon T \mid \Delta; \Theta$*, then* $\Sigma, \upsilon\colon A; \Gamma \vdash E[{}^{\upsilon}/_{\chi}]\colon T \mid \Delta; \Theta$ *holds.*

(3) *If* $\Sigma; \Gamma \vdash E\colon T \mid \Delta, a\colon A, b\colon A; \Theta$*, then* $\Sigma; \Gamma \vdash E[{}^{b}/_{a}]\colon T \mid \Delta, b\colon A; \Theta$ *holds.*

(4) *If* $\Sigma; \Gamma \vdash E\colon T \mid \Delta; \Theta, \alpha\colon A, \beta\colon A$*, then* $\Sigma; \Gamma \vdash E[{}^{\beta}/_{\alpha}]\colon T \mid \Delta; \Theta, \beta\colon A$ *holds.*

The types of expressions of $\lambda\mu^{S4}$ are preserved by reduction.

Proposition 1 (Subject Reduction). *If* $\Sigma; \Gamma \vdash E\colon T \,|\, \Delta; \Theta$ *and* $E \longrightarrow F$*, then* $\Sigma; \Gamma \vdash F\colon T \,|\, \Delta; \Theta$ *holds.*

Proof. This claim is shown by induction on the structure of one-step reduction with Lemma 6.

We will prove strong normalization of the $\lambda\mu^{S4}$-calculus. An expression is defined to be strongly normalizing if there does not exist any infinite reduction sequence starting from the expression.

Theorem 2 (Strong normalization of $\lambda\mu^{S4}$). *If* $\Sigma; \Gamma \vdash E : T \,|\, \Delta; \Theta$ *is derivable in* $\lambda\mu^{S4}$*, then* E *is strongly normalizing.*

We will show this theorem by giving translation from the $\lambda\mu^{S4}$-calculus into the $\lambda\mu$-calculus, and using strong normalization of the $\lambda\mu$-calculus. Strong normalization of the (second-order) $\lambda\mu$-calculus is already shown by Parigot [17].

Parigot's $\lambda\mu$-calculus [16] is given as follows.

Definition 8 (Parigot's $\lambda\mu$-calculus). Types (denoted by $\tau, \sigma, \ldots$) of the $\lambda\mu$-calculus are defined by:

Types $\quad \tau ::= X \mid \tau \supset \tau \mid \bot.$

We will write $\neg\tau$ as an abbreviation of $\tau \supset \bot$.

The $\lambda\mu$-calculus has λ-variables (denoted by $x, y, \ldots, \chi, \upsilon, \ldots$) and μ-variables (denoted by $a, b, \ldots, \alpha, \beta, \ldots$). We use distinguished μ-variables $\xi, \zeta, \ldots$ for type $\bot$. An expression (denoted by e) of the $\lambda\mu$-calculus is either an unnamed term (denoted by $t, u, \ldots$) or a named term (denoted by $n, m, \ldots$) defined by:

Expressions $\quad e ::= t \mid n$

Unnamed terms $t ::= x \mid \lambda x.t \mid tt \mid \mu a.n$,

Named terms $n ::= [a]t$.

A typing judgment of the $\lambda\mu$-calculus is either the form $\Gamma \vdash_{\lambda\mu} t \colon \tau \mid \Delta$ or the form $n \colon \Gamma \vdash_{\lambda\mu} \Delta$, where Γ is a set $x_1 : \tau_1, \ldots, x_n : \tau_n, \chi_1 : \sigma_1, \ldots, \chi_m : \sigma_m$ of λ-variable declarations, Δ is a set $a_1 : \tau'_1, \ldots, a_p : \tau'_p, \alpha_1 : \sigma'_1, \ldots, \alpha_q : \sigma'_q$ of μ-variable declarations. The μ-variable declarations of the type $\perp$ are not mentioned explicitly in typing judgments.

The typing rules of the $\lambda\mu$-calculus are given as follows.

$$\frac{}{\Gamma, x \colon \tau \vdash_{\lambda\mu} x \colon \tau \mid \Delta} \qquad \frac{\Gamma, x \colon \tau \vdash_{\lambda\mu} u \colon \sigma \mid \Delta}{\Gamma \vdash_{\lambda\mu} \lambda x.u \colon \tau \supset \sigma \mid \Delta} \qquad \frac{\Gamma \vdash_{\lambda\mu} t \colon \tau \supset \sigma \mid \Delta \quad \Gamma \vdash_{\lambda\mu} u \colon \tau \mid \Delta}{\Gamma \vdash_{\lambda\mu} tu \colon \sigma \mid \Delta}$$

$$\frac{\Gamma \vdash_{\lambda\mu} u \colon \tau \mid \Delta}{[a]u : \Gamma \vdash_{\lambda\mu} \Delta, a \colon \tau} \qquad \frac{n : \Gamma \vdash_{\lambda\mu} \Delta, a \colon \tau}{\Gamma \vdash_{\lambda\mu} \mu a.n \colon \tau \mid \Delta}.$$

We claim that $[\xi]u \colon \Gamma \vdash_{\lambda\mu} \Delta$ is derived from $\Gamma \vdash_{\lambda\mu} u \colon \perp \mid \Delta$, and $\Gamma \vdash_{\lambda\mu} \mu\xi.n \colon \perp \mid \Delta$ is derived from $n \colon \Gamma \vdash_{\lambda\mu} \Delta$, since the μ-variable declaration $\xi \colon \perp$ is not mentioned explicitly in typing judgments.

One step reduction $\rhd$ of the $\lambda\mu$-calculus is defined as the compatible closure of the following relations.

(Beta) $(\lambda x.u)t \rhd u[t/x]$

(Mu) $(\mu a.n)u \rhd \mu b.n[{}^{[b]\{-\}u}/_{[a]\{-\}}]$

(Rename) $[b]\mu a.n \rhd n[{}^{[b]\{-\}}/_{[a]\{-\}}]$

(Eta) $\mu a.[a]u \rhd u$ (a is not free in u)

$\rhd^+$ and $\rhd^*$ are defined as the transitive closure and the reflexive transitive closure of $\rhd$, respectively.

Theorem 3 (Strong normalization of $\lambda\mu$ (Parigot [17])). *Every typable expression is strongly normalizing in the $\lambda\mu$-calculus.*

Strictly speaking, strong normalization of the $\lambda\mu$-calculus without (Eta)-rule was shown by Parigot. Strong normalization of the system with (Eta)-rule is also shown immediately from Parigot's result.

Here we give a translation $(-)^{\mathrm{dn}}$ from the $\lambda\mu^{S4}$-calculus into the $\lambda\mu$-calculus. It maps each modal operator to double negation.

Definition 9 (Translation $(-)^{\mathrm{dn}}$). Let A be a normal type in the $\lambda\mu^{S4}$-calculus. The type $(A)^{\mathrm{dn}}$ of the $\lambda\mu$-calculus is defined by:

$(X)^{\mathrm{dn}} = X \qquad (A \supset B)^{\mathrm{dn}} = (A)^{\mathrm{dn}} \supset (B)^{\mathrm{dn}} \qquad (\neg A)^{\mathrm{dn}} = \neg(A)^{\mathrm{dn}}$

$(\Box A)^{\mathrm{dn}} = \neg\neg(A)^{\mathrm{dn}} \qquad (\Diamond A)^{\mathrm{dn}} = \neg\neg(A)^{\mathrm{dn}}$.

Let E be an expression of the $\lambda\mu^{S4}$-calculus. The expression $(E)^{\mathrm{dn}}_{\xi}$ of the $\lambda\mu$-calculus is defined by using a μ-variable ξ as follows.

$(x)^{\mathrm{dn}}_{\xi} = x \quad (\lambda x.M)^{\mathrm{dn}}_{\xi} = \lambda x.(M)^{\mathrm{dn}}_{\xi} \quad (MN)^{\mathrm{dn}}_{\xi} = (M)^{\mathrm{dn}}_{\xi}(N)^{\mathrm{dn}}_{\xi} \quad (\mu a.S)^{\mathrm{dn}}_{\xi} = \mu a.(S)^{\mathrm{dn}}_{\xi}$

$(\lambda x.S)^{\mathrm{dn}}_{\xi} = \lambda x.\mu\zeta.(S)^{\mathrm{dn}}_{\xi} \quad (M \cdot N)^{\mathrm{dn}}_{\xi} = [\xi]((M)^{\mathrm{dn}}_{\xi}(N)^{\mathrm{dn}}_{\xi}) \quad ([a]M)^{\mathrm{dn}}_{\xi} = [a](M)^{\mathrm{dn}}_{\xi}$

$(\chi)^{\mathrm{dn}}_{\xi} = \chi \qquad (\alpha M)^{\mathrm{dn}}_{\xi} = [\alpha](\lambda x.x(M)^{\mathrm{dn}}_{\xi})$

$(\Box M)^{\mathrm{dn}}_{\xi} = \lambda x.x(M)^{\mathrm{dn}}_{\xi} \qquad (\texttt{let}\ \Box\chi\ \texttt{be}\ M\ \texttt{in}\ S)^{\mathrm{dn}}_{\xi} = [\xi]((M)^{\mathrm{dn}}_{\xi}(\lambda\chi.\mu\zeta.(S)^{\mathrm{dn}}_{\xi}))$

$(\Diamond\alpha.S)^{\mathrm{dn}}_{\xi} = \mu\alpha.(S)^{\mathrm{dn}}_{\xi} \qquad (\texttt{dia}\langle x.S\rangle(M))^{\mathrm{dn}}_{\xi} = [\xi]((M)^{\mathrm{dn}}_{\xi}(\lambda x.\mu\zeta.(S)^{\mathrm{dn}}_{\xi}))$,

where ζ and ξ are different μ-variables.

We define $(\Gamma)^{\mathrm{dn}}$ by $x_1 \colon (A_1)^{\mathrm{dn}}, \ldots, x_n \colon (A_n)^{\mathrm{dn}}$ if Γ is a classical typing context $x_1 \colon A_1, \ldots, x_n \colon A_n$ in $\lambda\mu^{S4}$. We similarly define $(\Sigma)^{\mathrm{dn}}$ of a modal typing context Σ, and $(\Delta)^{\mathrm{dn}}$ of a classical typing cocontext Δ. We also define $\neg\neg(\Theta)^{\mathrm{dn}}$ by

$\alpha_1 : \neg\neg(B_1)^{\mathrm{dn}}, \ldots, \alpha_m : \neg\neg(B_m)^{\mathrm{dn}}$ if a modal cocontext Θ is $\alpha_1 : B_1, \ldots, \alpha_m : B_m$. Then the judgment $(J)^{\mathrm{dn}}_{\xi}$ of the $\lambda\mu$-calculus for a judgment J of the $\lambda\mu^{S4}$-calculus is given as follows:

$$(\Sigma; \Gamma \vdash M : A \mid \Delta; \Theta)^{\mathrm{dn}}_{\xi} = (\Sigma)^{\mathrm{dn}}, (\Gamma)^{\mathrm{dn}} \vdash_{\lambda\mu} (M)^{\mathrm{dn}}_{\xi} : (A)^{\mathrm{dn}} \mid (\Delta)^{\mathrm{dn}}, \neg\neg(\Theta)^{\mathrm{dn}},$$
$$(\Sigma; \Gamma \vdash S : \bot \mid \Delta; \Theta)^{\mathrm{dn}}_{\xi} = (S)^{\mathrm{dn}}_{\xi} : (\Sigma)^{\mathrm{dn}}, (\Gamma)^{\mathrm{dn}} \vdash_{\lambda\mu} (\Delta)^{\mathrm{dn}}, \neg\neg(\Theta)^{\mathrm{dn}}.$$

This translation preserves typing.

Proposition 2. *If J is derivable in $\lambda\mu^{S4}$, then $(J)^{\mathrm{dn}}_{\xi}$ is derivable in $\lambda\mu$.*

Proof. This claim is shown by induction on the derivation of the $\lambda\mu^{S4}$-calculus.

The translation satisfies the following property.

Lemma 8. (1) *$(E)^{\mathrm{dn}}_{\xi}[{}^{(N)^{\mathrm{dn}}_{\xi}}/_{x}] = (E[{}^{N}/_{x}])^{\mathrm{dn}}_{\xi}$ and $(E)^{\mathrm{dn}}_{\xi}[{}^{(N)^{\mathrm{dn}}_{\xi}}/_{\chi}] = (E[{}^{N}/_{\chi}])^{\mathrm{dn}}_{\xi}$ hold.*
(2) *$(E)^{\mathrm{dn}}_{\xi}[{}^{[\xi]\{-\}(N)^{\mathrm{dn}}_{\xi}}/_{[\alpha]\{-\}}] \rhd^{*} (E[{}^{N}/_{\alpha}])^{\mathrm{dn}}_{\xi}$ holds.*

Proof. (1) is shown by induction on the construction of E. (2) is also shown by induction on the construction of E.

Definition 10. Let $\mathcal{E}$ be an elimination context of the $\lambda\mu^{S4}$-calculus. Then $(\mathcal{E})^{\mathrm{dn}}_{\xi}$ is defined as follows: $([a]\{-\})^{\mathrm{dn}}_{\xi} = [a]\{-\}$, $([a]\{-\}N)^{\mathrm{dn}}_{\xi} = [a]\{-\}(N)^{\mathrm{dn}}_{\xi}$, $(\{-\} \cdot N)^{\mathrm{dn}}_{\xi} = [\xi]\{-\}(N)^{\mathrm{dn}}_{\xi}$, $(\mathtt{let}\ \Box\chi\ \mathtt{be}\ \{-\}\ \mathtt{in}\ S)^{\mathrm{dn}}_{\xi} = [\xi]\{-\}(\lambda\chi.\mu\zeta.(S)^{\mathrm{dn}}_{\xi})$, $(\mathtt{dia}\langle x.S\rangle(\{-\}))^{\mathrm{dn}}_{\xi} = [\xi]\{-\}(\lambda x.\mu\zeta.(S)^{\mathrm{dn}}_{\xi})$.

Then $(\mathcal{E})^{\mathrm{dn}}_{\xi}$ satisfies the following properties.

Lemma 9. (1) *$(\mathcal{E}\{M\})^{\mathrm{dn}}_{\xi} = (\mathcal{E})^{\mathrm{dn}}_{\xi}\{(M)^{\mathrm{dn}}_{\xi}\}$ holds.*
(2) *$(E)^{\mathrm{dn}}_{\xi}[{}^{(\mathcal{E})^{\mathrm{dn}}_{\xi}}/_{[a]\{-\}}] = (E[{}^{\mathcal{E}}/_{[a]\{-\}}])^{\mathrm{dn}}_{\xi}$ holds.*
(3) *$(\mathcal{E})^{\mathrm{dn}}_{\xi}\{\mu a.n\} \rhd^{+} n[{}^{(\mathcal{E})^{\mathrm{dn}}_{\xi}}/_{[a]\{-\}}]$ holds.*

Proof. (1) is shown by the case analysis of $\mathcal{E}$. (2) is shown by induction on E. (3) is shown by the case analysis of $\mathcal{E}$.

The translation $(-)^{\mathrm{dn}}_{\xi}$ maps each one-step reduction of the $\lambda\mu^{S4}$-calculus to one or more steps reduction of the $\lambda\mu$-calculus.

Proposition 3. *If $E \longrightarrow E'$ in $\lambda\mu^{S4}$, then $(E)^{\mathrm{dn}}_{\xi} \rhd^{+} (E')^{\mathrm{dn}}_{\xi}$ in $\lambda\mu$.*

Proof. The claim is proved by induction on the construction of $E \longrightarrow E'$ by using Lemmas 8 and 9. We show the cases of ($\beta\Box$), ($\beta\Diamond$), ($\mu\Box$), and ($\mu\Diamond$)-rules. The case of ($\beta\Box$) is shown as follows: $(\mathtt{let}\ \Box\chi\ \mathtt{be}\ \Box M\ \mathtt{in}\ S)^{\mathrm{dn}}_{\xi} = [\xi]((\Box M)^{\mathrm{dn}}_{\xi}(\lambda\chi.\mu\zeta.(S)^{\mathrm{dn}}_{\xi})) = [\xi]((\lambda x.x(M)^{\mathrm{dn}}_{\xi})(\lambda\chi.\mu\zeta.(S)^{\mathrm{dn}}_{\xi})) \rhd [\xi]((\lambda\chi.\mu\zeta.(S)^{\mathrm{dn}}_{\xi})(M)^{\mathrm{dn}}_{\xi}) \rhd [\xi]\mu\zeta.((S)^{\mathrm{dn}}_{\xi}[{}^{(M)^{\mathrm{dn}}_{\xi}}/_{\chi}]) = [\xi]\mu\zeta.(S[{}^{M}/_{\chi}])^{\mathrm{dn}}_{\xi} \rhd (S[{}^{M}/_{\chi}])^{\mathrm{dn}}_{\xi}$. The case of ($\beta\Diamond$) is shown as follows: $(\mathtt{dia}\langle x.R\rangle(\Diamond\alpha.S))^{\mathrm{dn}}_{\xi} = [\xi]((\Diamond\alpha.S)^{\mathrm{dn}}_{\xi}(\lambda x.\mu\zeta.(R)^{\mathrm{dn}}_{\xi})) = [\xi]((\mu\alpha.(S)^{\mathrm{dn}}_{\xi})(\lambda x.R)^{\mathrm{dn}}_{\xi})) \rhd [\xi]\mu\zeta.(S)^{\mathrm{dn}}_{\xi}[{}^{[\zeta]\{-\}(\lambda x.R)^{\mathrm{dn}}_{\xi}}/_{[\alpha]\{-\}}] \rhd (S)^{\mathrm{dn}}_{\xi}[{}^{[\xi]\{-\}(\lambda x.R)^{\mathrm{dn}}_{\xi}}/_{[\alpha]\{-\}}] \rhd^{*} (S[{}^{\lambda x.R}/_{\alpha}])^{\mathrm{dn}}_{\xi}$. The rules ($\mu\Box$) and ($\mu\Diamond$) are written together by $\mathcal{E}\{\mu a.S\} \longrightarrow S[{}^{\mathcal{E}}/_{[a]\{-\}}]$, where $\mathcal{E}$ is an elimination contexts. Then these cases are shown using by Lemma 9 as follows: $(\mathcal{E}\{\mu a.S\})^{\mathrm{dn}}_{\xi} = (\mathcal{E})^{\mathrm{dn}}_{\xi}\{(\mu a.S)^{\mathrm{dn}}_{\xi}\} = (\mathcal{E})^{\mathrm{dn}}_{\xi}\{\mu a.(S)^{\mathrm{dn}}_{\xi}\} \rhd^{+} (S)^{\mathrm{dn}}_{\xi}[{}^{(\mathcal{E})^{\mathrm{dn}}_{\xi}}/_{[a]\{-\}}] = (S[{}^{\mathcal{E}}/_{[a]\{-\}}])^{\mathrm{dn}}_{\xi}$.

We complete the proof of strong normalization of the $\lambda\mu^{S4}$-calculus.

Proof (Theorem 2). Assume that E is typable in $\lambda\mu^{S4}$ and there is an infinite reduction sequence $E \longrightarrow E_1 \longrightarrow \ldots$ starting from E. Then $(E)^{\mathrm{dn}}_{\xi} \rhd^{+} (E_1)^{\mathrm{dn}}_{\xi} \rhd^{+} (E_2)^{\mathrm{dn}}_{\xi} \rhd^{+} \ldots$ is an infinite reduction sequence starting from $(E)^{\mathrm{dn}}_{\xi}$ by Proposition 3. Since $(E)^{\mathrm{dn}}_{\xi}$ is typable in $\lambda\mu$ by proposition 2, it contradicts Theorem 3.

Finally, we will show confluence of the $\lambda\mu^{S4}$-calculus.

Proposition 4 (Local confluence of $\lambda\mu^{S4}$). *If $E \longrightarrow E_1$ and $E \longrightarrow E_2$, then there exists E_3 that satisfies $E_1 \longrightarrow^{*} E_3$ and $E_2 \longrightarrow^{*} E_3$ for any expressions E, E_1, and E_2 of $\lambda\mu^{S4}$.*

Proof. This claim is shown by induction on the structure of E.

Confluence of the $\lambda\mu^{S4}$-calculus is immediately shown by using Theorem 2, Proposition 4, and Newman's lemma [14].

Theorem 4 (Confluence of $\lambda\mu^{S4}$). *If $E \longrightarrow^{*} E_1$ and $E \longrightarrow^{*} E_2$, then there exists E_3 that satisfies $E_1 \longrightarrow^{*} E_3$ and $E_2 \longrightarrow^{*} E_3$ for any expressions E, E_1, and E_2 of $\lambda\mu^{S4}$.*

5 Discussions

(1) Syntax sugars. We define an additional term $\mathtt{let}\ \Box\chi\ \mathtt{be}\ M\ \mathtt{in}\ N$ as an abbreviation of $\mu a.\mathtt{let}\ \Box\chi\ \mathtt{be}\ M\ \mathtt{in}\ [a]N$. This term validates the following rules:

$$\frac{\Sigma, \chi : A; \Gamma \vdash N : B \mid \Delta; \Theta \quad \Sigma; \Gamma \vdash M : \Box A \mid \Delta; \Theta}{\Sigma; \Gamma \vdash \mathtt{let}\ \Box\chi\ \mathtt{be}\ M\ \mathtt{in}\ N : B \mid \Delta; \Theta}, \text{ and } \mathtt{let}\ \Box\chi\ \mathtt{be}\ \Box M\ \mathtt{in}\ N \longrightarrow^{+} N[M/\chi].$$

We also define an additional statement $\langle\alpha\rangle M$ by $\mathtt{dia}\langle x.\alpha x\rangle(M)$. It validates the following rules:

$$\frac{\Sigma; \Gamma \vdash M : \Diamond A \mid \Delta; \Theta}{\Sigma; \Gamma \vdash \langle\alpha\rangle M : \bot \mid \Delta; \Theta, \alpha : A}, \text{ and } \langle\beta\rangle\Diamond\alpha.S \longrightarrow^{+} S\,[{}^{\beta}/_{\alpha}].$$

(2) (η)-rules for $\Box$ and $\Diamond$ types. In this paper, we gave only (β) and (μ)-rules for $\Box$ and $\Diamond$ types, since we started from the normalization procedure of CND^{S4}. We may define (η)-rules for $\Box$ and $\Diamond$-operators by:

$(\eta\Box)$ $\mathtt{let}\ \Box\chi\ \mathtt{be}\ M\ \mathtt{in}\ \Box\chi \longrightarrow M$ (a is not free in M),
$(\eta\Diamond)$ $\Diamond\alpha.\langle\alpha\rangle M \longrightarrow M$ (α is not free in M).

Unfortunately, $(\eta\Box)$ breaks confluency of $\lambda\mu^{S4}$. For example, $[a]\mathtt{let}\ \Box\chi\ \mathtt{be}\ M\ \mathtt{in}\ \Box\chi$ is reduced to $\mathtt{let}\ \Box\chi\ \mathtt{be}\ M\ \mathtt{in}\ [a](\Box\chi)$ by (rn)-rule, and is also reduced to $[a]M$ by $(\eta\Box)$-rule.

(3) Computational interpretation of $\lambda\mu^{S4}$. Finally, we try to give a computational interpretation of the $\lambda\mu^{S4}$-calculus. The $\lambda\mu^{S4}$-calculus is an extension of the modal λ-calculus [4,6] without (η)-rules. Davies and Pfenning [4] showed the modal λ-calculus provides a framework for studying computation in stages. A value of type $\Box A$ is considered as a program which can be used at *any* later stages. Thus they interpreted a type $\Box A$ as a type of program codes of type A. By taking the dual statement, a type $\Diamond A$ will be interpreted as a type of programs that can be used at *some* later stage.

A judgment $\overline{\chi : A}; \overline{x : B} \vdash E : T \mid \overline{a : C}; \overline{\alpha : D}$ of $\lambda\mu^{S4}$ will be interpreted as follows: If each modal variable χ is supplied a program code of type A, and each classical variable

x is supplied a value of type B, then evaluation of the expression E will either return a value of type T, or pass a value of type C to some classical variable a, or pass a program of type D that will be used at some later stage to some modal variable α.

Each expression for possibility operator is interpreted as follows. A statement αM passes the value of M to α. A term $\Diamond\alpha.S$ returns a value which is passed to α in S. These interpretations are similar to those of $[a]M$ and $\mu a.S$. The different point is that the returned value of $\Diamond\alpha.S$ is used at some later stage though the returned value of $\mu a.S$ is used at the current stage. A statement $\mathtt{dia}\langle x.R\rangle(M)$ receives the output from M at some later stage, and passes it to the continuation $\lambda x.R$. The continuation $\mathtt{dia}\langle x.R\rangle(-)$ is understood as a package of the continuation $\lambda x.R$, and keeps waiting for input values exceeding stages. We call this a *persistent continuation* as the dual counterpart of persistent code [24].

(4) Staged computation with exception handling. As a possible application of persistent continuations, we give an example of staged computation with exception handling. We will informally assume the call-by-value $\lambda\mu^{S4}$ with a recursion operator `fix`, **if-then-else** expression, and the types `int` (integers) and `list` (lists of integers). By using expressions of the $\lambda\mu$-calculus, exception operators `catch` and `throw` are represented as follows: $\mathtt{catch}\ a.M := \mu a.[a]M$, and $\mathtt{throw}\ (a, M) := \mu b.[a]M$. For example, let us consider the following program with `catch` and `throw` operators.

```
mlist = λN.λL.catch a.(mul N L)
mul = fix λF.λN.λL. if N = 0 then 1 else
                    if L = nil then 1 else
                    if hd(L) = 0 then throw (a, 0) else hd(L) ∗ (F (N − 1) tl(L)).
```

The function `mlist` of type $\mathtt{int} \supset \mathtt{list} \supset \mathtt{int}$ takes an integer N and a list L as its inputs, and `mul` recursively multiplies the first N elements of L. If `mul` encounters an element 0 during the calculation, then $\mathtt{throw}\ (a, 0)$ throws exception to `catch` operator in `mlist`, and `mlist` immediately returns 0. However, we cannot write a staged program using `catch` and `throw` that generates a program code of ($\mathtt{mlist}\ N$) when the argument N is statically known, because `catch` and `throw` must be used in the same scope of a $\Box$-operator. For example, the program $\mathtt{catch}\ a.\Box(\mathtt{throw}\ (a, 0))$ is not valid, since the classical covariable a occurs freely in the scope of $\Box$-operator.

On the other hand, the following new operators can be defined in $\lambda\mu^{S4}$:

$$\mathtt{catch}_\Diamond \alpha.M := \Diamond\alpha.\langle\alpha\rangle M, \quad \text{and} \quad \mathtt{throw}_\Diamond(\alpha, M) := \Diamond\beta.\langle\alpha\rangle M \qquad (\beta \text{ is not free in } M).$$

They operate similar to `catch` and `throw` by assuming $(\eta\Diamond)$-rule:

$$\mathtt{catch}_\Diamond \alpha.V \longrightarrow^* V \qquad (\alpha \text{ is not free in } V, \text{ and } V \text{ is a value}),$$
$$\mathtt{catch}_\Diamond \alpha.(\mathtt{throw}_\Diamond(\alpha, M)) \longrightarrow^* M \qquad (\alpha \text{ is not free in } M),$$
$$\Diamond\beta.\mathtt{dia}\langle x.R\rangle(\mathtt{throw}_\Diamond(\alpha, M)) \longrightarrow^* \mathtt{throw}_\Diamond(\alpha, M).$$

We claim that $\mathtt{catch}_\Diamond$ and $\mathtt{throw}_\Diamond$ can be used if they are *not* in the same scope of a $\Box$-operator. For example, the program $\mathtt{catch}_\Diamond \alpha.\Box(\mathtt{throw}_\Diamond(\alpha, 0))$ is a valid program, since the modal covariable α can occur freely in the scope of $\Box$-operator. We also define a term $\mathtt{cast}(M)$ by $\Diamond\alpha.\alpha M$, where α is not free in M. The meaning of $\mathtt{cast}(M)$ is that the value of M will be used at some later stage. This term takes out the inside continuation by unpacking a persistent continuation, and passes the value of M to the inside continuation. Thus, $\mathtt{dia}\langle x.R\rangle(\mathtt{cast}(M))$ is reduced to $(\lambda x.R)M$.

We then define the function $\texttt{mlist}_S$ of type $\texttt{int} \supset \Box(\texttt{list} \supset \Diamond\texttt{int})$ as follows.

$$\texttt{mlist}_S(N) = \texttt{let}\ \Box\chi\ \texttt{be}\ (\texttt{mul}_S N)\ \texttt{in}\ \Box(\lambda L.\texttt{catch}_\Diamond \alpha.(\chi L)),$$

$$\texttt{mul}_S = \texttt{fix}\ \lambda F.\lambda N.\ \texttt{if}\ N = 0\ \texttt{then}\ \Box(\lambda L.\texttt{cast}(1))\ \texttt{else}\ \texttt{let}\ \Box\chi\ \texttt{be}\ F(N-1)\ \texttt{in}\ \Box P(\chi),$$

$$P(\chi) = \lambda L.\texttt{if}\ L = \texttt{nil}\ \texttt{then}\ \texttt{cast}(1)\ \texttt{else}$$
$$\qquad \texttt{if}\ \texttt{hd}(L) = 0\ \texttt{then}\ \texttt{throw}_\Diamond(\alpha, \texttt{cast}(0))\ \texttt{else}\ \ \Diamond\beta.\texttt{dia}\langle x.\beta(\texttt{hd}(L) * x)\rangle(\chi\ \texttt{tl}(L)).$$

The term $\texttt{mlist}_S(n)$ generates a program code of type $\Box(\texttt{list} \supset \Diamond\texttt{int})$ that calculates the multiplication of the first n elements of the input list. The term $(\texttt{mul}_S n)$ is reduced to $\Box(P^n(\lambda L.\texttt{cast}(1)))$, where $P^n(M)$ is $P(\ldots P(M)\ldots)$ (n times of P). Thus the term $\texttt{mlist}_S(n)$ is reduced to $\Box(\lambda L.\texttt{catch}_\Diamond \alpha.P^n(\lambda L'.\texttt{cast}(1))L)$. Hence $\texttt{unbox}(\texttt{mlist}_S(4))[2,4,1,3,5]$ is reduced to $\texttt{catch}_\Diamond \alpha.\Diamond\beta.\texttt{dia}\langle\, x.\beta(2 * 4 * 1 * 3 * x)\,\rangle(\texttt{cast}(1))$, and then $\texttt{cast}(24)$ is obtained. where $\texttt{unbox}(M)$ is $\texttt{let}\ \Box\chi\ \texttt{be}\ M\ \texttt{in}\ \chi$. On the other hand, $\texttt{unbox}(\texttt{mlist}_S(4))[2,4,0,3,5]$ is reduced to $\texttt{catch}_\Diamond \alpha.\Diamond\beta.\texttt{dia}\langle\, x.\beta(2{*}4{*}x)\,\rangle(\texttt{throw}_\Diamond(\alpha, \texttt{cast}(0)))$, and then it reduced to $\texttt{cast}(0)$ by $\texttt{catch}_\Diamond/\texttt{throw}_\Diamond$mechanism. This simulates the `catch`/`throw` mechanism in `mlist`.

6 Conclusion and Future Work

We proposed a new natural deduction system CND^{S4} for classical S4 modal logic. This system was an extension of Parigot's Classical Natural Deduction for classical logic. We then introduced the $\lambda\mu^{S4}$-calculus as a computational extraction of CND^{S4}, and showed subject reduction, confluency, and strong normalization. In the previous section, we discussed computational interpretation of the possibility operator introducing the notion of persistent continuation. As we observed, the possibility operator enabled the necessity operator to provide a theoretical framework for staged computation with exception handling.

Our future work is as follows. (1) The call-by-value $\lambda\mu^{S4}$-calculus: The calculus given in this paper is based on call-by-name. We can also give the call-by-value variant of $\lambda\mu^{S4}$-calculus, which is informally considered in Section 5. CPS based analysis is deeply related to call-by-value systems. It will give us a new approach for studying computational aspect of classical modal logic. (2) Formulation and application of persistent continuations: For the clearer understanding of the possibility operator, persistent continuations should be explored more deeply in the further work. (3) Computational duality in classical modal logic: The $\lambda\mu^{S4}$-calculus is given by adding the necessity and possibility operators to the $\lambda\mu$-calculus in a symmetric way. This means that the duality between call-by-value and call-by-name of classical logic will be naturally extended to classical modal logic. This expected result will give us an approach for studying persistent continuations.

References

1. Bierman, G.M., de Paiva, V.: On an intuitionistic modal logic. Studia Logica 65(3), 383–416 (2000)
2. Curien, P.L., Herbelin, H.: The duality of computation. In: Proceedings of the 5th ACM SIGPLAN International Conference on Functional Programming, ICFP, pp. 233–243 (2000)
3. Davies, R.: A temporal-logic approach to binding-time analysis. In: Proceedings of 11 th Annual IEEE Symposium on Logic in Computer Science, pp. 184–195. IEEE Computer Society Press, Los Alamitos (1996)

4. Davies, R., Pfenning, F.: A modal analysis of staged computation. Journal of the ACM 48(3), 555–604 (2001)
5. de Groote, P.: On the relation between the lambda-mu-calculus and the syntactic theory of sequential control. In: Pfenning, F. (ed.) LPAR 1994. LNCS, vol. 822, pp. 31–43. Springer, Heidelberg (1994)
6. Ghani, N., de Paiva, V., Ritter, E.: Explicit Substitutions for Constructive Necessity. In: Larsen, K.G., Skyum, S., Winskel, G. (eds.) ICALP 1998. LNCS, vol. 1443, pp. 743–754. Springer, Heidelberg (1998)
7. Griffin, T.G.: A formulae-as-types notion of control. In: Proc. of the 1990 Principles of Programming Languages Conference, pp. 47–58. IEEE Computer Society Press, Los Alamitos (1990)
8. Groote, P.D.: An environment machine for the $\lambda\mu$-calculus. Mathematical Structures in Computer Science 8(6), 637–669 (1998)
9. Kakutani, Y.: Duality between Call-by-Name Recursion and Call-by-Value Iteration. In: Bradfield, J.C. (ed.) CSL 2002. LNCS, vol. 2471, pp. 506–521. Springer, Heidelberg (2002)
10. Kakutani, Y.: Call-by-Name and Call-by-Value in Normal Modal Logic. In: Shao, Z. (ed.) APLAS 2007. LNCS, vol. 4807, pp. 399–414. Springer, Heidelberg (2007)
11. Kimura, D.: Duality between Call-by-value Reductions and Call-by-name Reductions. IPSJ Journal 48(4), 1721–1757 (2007)
12. de Paz, M., Medeiros, N.: A new S4 classical modal logic in natural deduction. Journal of Symbolic Logic 71(3), 799–809 (2006)
13. Nanevski, A.: A Modal Calculus for Exception Handling. In: The 3rd intuitionistic modal logics and applications workshop (2005)
14. Newman, M.H.A.: On theories with a combinatorial definition of "equivalence". Annals of Mathematics 43(2), 223–243 (1942)
15. Ong, C.-H.L., Stewart, C.A.: A Curry-Howard foundation for functional computation with control. In: Proc. of the Symposium on Principles of Programming Languages, pp. 215–227 (1997)
16. Parigot, M.: $\lambda\mu$-calculus: an algorithmic interpretation of classical natural deduction. In: Voronkov, A. (ed.) LPAR 1992. LNCS, vol. 624, pp. 190–201. Springer, Heidelberg (1992)
17. Parigot, M.: Strong normalization for second order classical natural deduction. In: Proceedings of Eighth Annual IEEE Symposium on Logic in Computer Science, pp. 39–46 (1993)
18. Pfenning, F., Davies, R.: A judgmental reconstruction of modal logic. Mathematical Structures in Computer Science 11, 511–540 (2001)
19. Prawitz, D.: Natural Deduction: A Proof-Theoretical Study. Almqvist and Wiksell, Stockholm (1965)
20. Selinger, P.: Control Categories and Duality: on the Categorical Semantics of the Lambda-Mu Calculus. Mathematical Structures in Computer Science, 207–260 (2001)
21. Shan, C.-C.: A Computational Interpretation of Classical S4 Modal Logic. In: The 3rd intuitionistic modal logics and applications workshop (2005)
22. Taha, W., Sheard, T.: MetaML and multi-stage programming with explicit annotations. Theoretical Computer Science 248(1-2), 211–242 (2000)
23. Wadler, P.: Call-by-Value is Dual to Call-by-Name – Reloaded. In: Giesl, J. (ed.) RTA 2005. LNCS, vol. 3467, pp. 185–203. Springer, Heidelberg (2005)
24. Yuse, Y., Igarashi, A.: A modal type system for multi-level generating extensions with persistent code. In: Proceedings of the 8th ACM SIGPLAN Symposium on Principles and Practice of Declarative Programming, pp. 201–212 (2006)