Upper and Lower Bounds of Space Complexity of Self-Stabilizing Leader Election in Mediated Population Protocol

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Abstract. This paper investigates the space complexity of a self stabilizing leader election in a mediated population protocol (SS-LE MPP). Cai, Izumi and Wada (2009) showed that SS-LE in a population protocol (SS-LE PP) for \( n \) agents requires at least \( n \) agent-states, and gave a SS-LE PP with \( n \) agent-states for \( n \) agents. MPP is a model of distributed computation, introduced by Chatzigiannakis, Michail and Spirakis (2009) as an extension of PP allowing an extra memory on every agents pair. While they showed that MPP is stronger than PP in general, it was not known if a MPP can really reduce the space complexity of SS-LE with respect to agent-states. We in this paper give a SS-LE MPP with \( (2/3)n \) agent-states and a single bit memory on every agents pair for \( n \) agents. We also show that there is no SS-LE MPP with any constant agent-states and any constant size memory on each agents-pair for general \( n \) agents.

Keywords: Mobile agents, anonymous, population protocols, self-stabilization, leader election.

1 Introduction

Population Protcol (PP), proposed by Angluin et al. [1], is a model of distributed computation consisting of agents and communication links among them, and Mediated Population Protocol (MPP) proposed by Chatzigiannakis et al. [7], is an extended model of PP allowing memories on communication links. PP and MPP are models of sensor networks consisting of passively mobile agents with limited computational resources, motivated by practical networks such as networks of smart sensors attached to cars or animals, synthesis of chemical materials, complex biosystems, and so on (cf. [17]).

In MPP, every agent is identically programmed as a finite state machine, and every communication link is equipped with a (finite) buffer. The agents sequentially interact with each other updating their states; a pair of agents chosen by a scheduler updates their own agent-states and edge-states between them in an interaction. The order of interactions of agent-pairs is unpredictable, and is
scheduled by an adversarial scheduler satisfying a *fairness condition*; the scheduler must accept any possible interaction within a finite time if a configuration in which the interaction can arise should appear infinitely many times.

Angluin et al. [3] discussed the *leader election* in a population protocol, which is a fundamental problem in distributed computing, and introduced the problem of *self stabilizing leader election* in a population protocol (SS-LE PP, for short). In a SS-LE PP, any initial configuration of agent-states eventually have to reach at a configuration whose successive configurations contain exactly one leader. Thus a SS-LE PP should be equipped with seemingly conflicting functions; the protocol has to decrease the number of leaders if a configuration contains two or more leaders, while the protocol has to appoint an agent to be a leader if a configuration does not contain a leader. This causes some difficulties on SS-LE, as it is usual with self-stabilizing distributed problems.


This paper is concerned with the space complexity of *self stabilizing leader election* in a *mediated population protocol* (SS-LE MPP, for short) for $n$ agents, where we assume that an interaction graph is complete as did Cai, Izumi and Wada [5]. We present a SS-LE MPP with roughly $(2/3)n$ agent-states and two edge-states for $n$ agents. We also show that there is no SS-LE MPP with any constant agent-states and any constant edge-states for general $n$ agents. As far as the authors know, this is the first result on SS-LE MPP. One may say it obvious that the number of agent-stats decreases in MPP comparing with PP due to extra memories on edges. In fact, it is clear that $n$ is also sufficient for the number of agent-states in SS-LE MPP for $n$ agents. However, extra memories on edges in MPP, which are expected to resolve the issue of conflicting functions in the self-stabilizing setting instead of a certain number of agent-states, may cause another issue of increasing possible (bad) initial configurations in the self-stabilizing setting.

This paper is organized as follows; in Section 2, we describe the detail of our model. To explain our basic idea for reducing agent-states, we in Section 3 give a SS-LE MPP with $n - 1$ agent-states and 2 edge-states for $n$ agents. In Section 4, we present a SS-LE MPP with $(2/3)n$, $\lceil(2/3)n\rceil + 1$ in precise, agent-states and two edge-states for $n$ agents. In Section 5 we give lower bounds of agent-states of a SS-LE MPP.

## 2 Model Description — SS-LE MPP

A *mediated population protocol* is defined by 3-tuple $(Q, S, \delta)$, where $Q$ denotes a finite set of agent-states, $S$ denotes a finite set of edge states, and $\delta : Q \times Q \times S \rightarrow$
\(Q \times Q \times S\) denotes a transition function. Let \(A\) denote the set of anonymous agents and let \(n = |A|\), and let \(\mathcal{C} \overset{\text{def}}{=} Q^A \times S^{|A|}\) denote all configurations. A transition from a configuration \(C \in \mathcal{C}\) to the next configuration \(C' \in \mathcal{C}\) is defined as follows. An arbitrary pair of agents \(a_i, a_j \in A(a_i \neq a_j)\) is chosen by a scheduler, thus an interaction graph is complete in our model. States of the agents \(a_i\) and \(a_j\), and a state of an edge \(\{a_i, a_j\}\) are updated according to a transition function \(\delta\). Let \(r : (p, q, s) \mapsto (p', q', s')\) denote a specified transition rule of \(\delta\), and let \(C \xrightarrow{r; a_i, a_j} C'\) denote a transition from \(C \in \mathcal{C}\) to \(C' \in \mathcal{C}\) in which agents \(a_i\) and \(a_j\) interact and their states \(p, q\) and edge-state \(s\) between them are updated to \(p', q', s'\) according to the rule \(r\) of \(\delta\). We simply write \(C \xrightarrow{r} C'\) without confusing. An execution of a protocol is represented by an infinite sequence of configurations and transitions \(C_0, r_0, C_1, r_1, \ldots\), where \(C_0\) is an initial configuration and \(C_i \xrightarrow{r_i} C_{i+1}(i \geq 0)\).

We assume that a scheduler in a MPP is adversarial but (globally) fair, as usual (cf. [5]). Thus we have to think that an adversarial scheduler schedules the order of interactions in a worst case scenario for us, but it is forced to satisfy that if a configuration \(C \in \mathcal{C}\) appears infinitely often in an execution, a configuration \(C' \in \mathcal{C}\) must also appear infinitely often in an execution, where \(C'\) is a configuration obtained by an arbitrary transition \(r \in \delta\) which arises in \(C\). We say that \(C\) eventually transits to \(C'\), denoted by \(C \xrightarrow{\star} C'\), if \(C'\) must appear after \(C\) by the adversarial but globally fair scheduler in MPP. In addition, we describe a sequence of transitions as the trace \(T\).

Leader election in a MPP is to assign a special state, representing a “leader”, in \(S\) to exactly one agent. We say a configuration \(C \in \mathcal{C}\) is legal if \(C\) contains exactly one agent with the leader state, and so does any configuration \(C'\) satisfying \(C \xrightarrow{\star} C'\). Let \(\mathcal{L}\) denote the set of all legal configurations. We say a protocol for the leader election (for a distributed problem, in general), is self-stabilizing if \(C \xrightarrow{\star} C', C' \in \mathcal{L}\) hold for any \(C \in \mathcal{C}\). We simply say SS-LE MPP as a mediated population protocol for the leader election which is self stabilizing.

Our goal is to give upper and lower bounds of the sizes of the agent-states \(Q\) and edge-states \(S\) for SS-LE MPP concerning the number of agents \(n\). Main results of the paper are to give a SS-LE MPP with \(|Q| = \lfloor(2/3)n\rfloor + 1\) and \(|S| = 2\) for \(n\) agents in Section 4, and to show that there is no SS-LE MPP with constant sizes of \(Q\) and \(S\) for general \(n\) agents in Section 5. To describe our basic idea for reducing the number of agent-states, we in Section 3 give a SS-LE MPP with \(|Q| = n - 1\) and \(|S| = 2\) for \(n\) agents.

3 Simple SS-LE MPP with \(n - 1\) Agent-States

In this section, we show the following.

**Theorem 1.** There exists a SS-LE MPP with \(n - 1\) agent-states and 2 edge-states for \(n(\geq 4)\) agents.

We give a constructive proof. In particular, we show that Protocol \(P_1\), defined as follows, is a SS-LE MPP.
Protocol \( P_1 \)

\[ Q = \{q_0, q_1, \ldots, q_{n-2}\}, \text{ where } q_0 \text{ denotes the leader state.} \]

\[ S = \{s_0, s_1\}, \]

\[ \delta = \{ \]

\[ r_1: (q_0, q_0, s) \mapsto (q_0, q_{n-2}, s_0) \quad \text{for } s \in S, \]

\[ r_2: (q_1, q_1, s) \mapsto (q_1, q_2, s_0) \quad \text{for } s \in S, \]

\[ r_3: (q_2, q_2, s_0) \mapsto (q_2, q_2, s_1), \]

\[ r_4: (q_2, q_2, s_1) \mapsto (q_2, q_1, s_0), \]

\[ r_5: (q_2, q_1, s_1) \mapsto (q_2, q_0, s_0) \quad ((q_1, q_2, s_1) \mapsto (q_0, q_2, s_0), \text{ symmetrically}), \]

\[ r_6: (q_i, q_i, s) \mapsto (q_i, q_{i-1}, s_0) \quad \text{for } i \geq 3, s \in S, \]

\[ r_7: (q_j, q_k, s) \mapsto (q_j, q_k, s_0) \quad \text{for } j \neq k, s \in S, \text{ except for the case of } r_5. \}

\]

**Remark.** Except for Transition \( r_5 \), the state of an agent can change only when the agent interacts with another agent in the same state.

Let \( \gamma_k(C) \) for \( k \in \{0, 1, \ldots, n - 2\} \) denote the number of agents with state \( q_k \) in a configuration \( C \in \mathcal{C} \). We define a set of configurations \( \mathcal{L} \subset \mathcal{C} \) by

\[ \mathcal{L} \overset{\text{def}}{=} \left\{ C \in \mathcal{C} \middle| \begin{array}{l}
\gamma_k(C) > 0 \text{ for } k \in \{0, 1, \ldots, n - 2\}, \\
\gamma_1(C) + \gamma_2(C) = 3, \\
\text{both ends of an edge with state } s_1 \text{ are agents with state } q_2.
\end{array} \right\}. \]

Note that the number of edges with state \( s_1 \) in \( C \in \mathcal{L} \) is at most one since \( \gamma_2(C) \) is at most two from the definition of \( \mathcal{L} \).

In the following, we claim that \( \mathcal{L} \) is the set of legal configurations for Protocol \( P_1 \). Let \( H \) denote a subconfiguration of \( C \in \mathcal{L} \) consisting of three agents with states \( q_1 \) or \( q_2 \) and three edges among them. Then \( H \) can be one of three types of subconfigurations \( H_1, H_2, H_3 \) of six possible types \( H_1, H_2, H_3, H_4, H_5, H_6 \) in Fig.[1] which satisfy that the number of edges with state \( s_1 \) is at most one, \( \gamma_1(C) > 0 \) and \( \gamma_2(C) > 0 \). First, we show that \( \mathcal{L} \) is “closed” under the transition function \( \delta \).

**Lemma 2.** If configurations \( C \) and \( C' \) satisfy \( C \in \mathcal{L} \) and \( C \xrightarrow{r} C' \), then \( C' \in \mathcal{L} \) and \( C' \xrightarrow{r} C \) hold.

**Proof.** Transition \( r_1 \) cannot arise in a configuration \( C \in \mathcal{L} \) by the condition \( \gamma_0(C) = 1 \) in \( \mathcal{L} \). By the condition \( \gamma_1(C) + \gamma_2(C) = 3 \) and \( \gamma_k(C) > 0 \) for \( k \in \{0, 1, \ldots, n - 2\} \), \( C \) satisfies \( \gamma_k(C) = 1 \) for \( k \in \{3, 4, \ldots, n - 2\} \). Thus, Transition \( r_6 \) cannot arise in \( C \in \mathcal{L} \).

Now we show that Transition \( r_5 \) cannot arise in \( C \). Since the edge-state \( s_1 \) appears only within the subconfiguration \( H \) of \( C \in \mathcal{L} \), it is enough to show that Transition \( r_5 \) cannot arise in subconfigurations \( H_1, H_2, H_3 \) of \( C \).

**Case 1.** \( H = H_1 \): Consider an agent with state \( q_1 \) as \( a \) and agents with state \( q_2 \) as \( b, c \), and consider every state of every edge among them as state \( s_0 \). Then Transitions \( r_3 \) or \( r_7 \) can arise in \( H_1 \), that is \( H_1 \xrightarrow{r_3} H_2 \) or \( H_1 \xrightarrow{r_7} H_1 \).
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Case 2. \( H = H_2 \): Consider an agent with state \( q_1 \) as \( a \) and agents with state \( q_2 \) as \( b, c \), and consider a state of the only one edge between the two agents \( b, c \) as state \( s_1 \). Then Transitions \( r_4 \) or \( r_7 \) can arise in \( H_2 \), that is \( H_2 \overset{r_4}{\rightarrow} H_3 \) or \( H_2 \overset{r_7}{\rightarrow} H_2 \).

Case 3. \( H = H_3 \): Consider agents with state \( q_1 \) as \( a, b \) and an agent with state \( q_2 \) as \( c \) and consider every state of every edge among them as state \( s_0 \). Then Transitions \( r_2 \) or \( r_7 \) can arise in \( H_3 \), that is \( H_3 \overset{r_2}{\rightarrow} H_1 \) or \( H_3 \overset{r_7}{\rightarrow} H_3 \).

Therefore, if \( C \in \mathcal{L} \) and \( C' \) satisfies \( C \overset{*}{\rightarrow} C' \), then \( C' \in \mathcal{L} \) and \( C' \overset{*}{\rightarrow} C \) hold. □

Next, we show that any configuration \( C \in \mathcal{C} \) eventually transits to a configuration \( C' \in \mathcal{L} \), in Lemma 5. To show Lemma 5, we show Lemmas 3 and 4.

**Lemma 3.** If a configuration \( C \in \mathcal{C} \) satisfies \( \gamma_k(C) > 0 \) for \( k \in \{0, 2, 3, \ldots, n - 2\} \), and \( C \overset{*}{\rightarrow} C' \), then the configuration \( C' \) also satisfies \( \gamma_k(C') > 0 \).

**Proof.** After an agent with state \( q_k \) for \( k \in \{0, 2, 3, \ldots, n - 2\} \) interacts with any other agent, \( \gamma_k(C) \) decreases at most one in any transition. In fact, \( \gamma_k(C) \) decreases only when the agent interacts with another agent in the same state \( q_k \). This implies that \( \gamma_k(C) \) never decreases from one to zero by any transition. □

**Lemma 4.** If configurations \( C, C' \in \mathcal{C} \) satisfy \( \gamma_0(C) = 0 \), \( C \overset{*}{\rightarrow} C' \) and \( \gamma_0(C') = 0 \), then the followings hold:

1. \( \sum_{i=1}^{k} \gamma_i(C') \geq \sum_{i=1}^{k} \gamma_i(C) \) for any \( k \in \{2, 3, \ldots, n - 2\} \).
2. If \( \gamma_i(C) > 0 \), then \( \gamma_i(C') > 0 \).
Proof. 1. If $\gamma_0(C) = 0$ and $\gamma_0(C') = 0$ hold, Transition $r_5$ cannot have arisen on $C \xrightarrow{r} C'$. Note that Transition $r_7$ does not change any agent-state. Since $n - 2$ states are assigned to $n$ agents, there exists a pair of agents and they are in a common state $q_i$. When $i \geq 3$, Transitions $r_6$ or $r_7$ can arise in $C$ and exactly one of the agents changes its state from $q_i$ to $q_{i-1}$. Thus $\gamma_{i-1}(C') + \gamma_i(C') = \gamma_{i-1}(C) + \gamma_i(C)$ and $\gamma_{i-1}(C') = \gamma_{i-1}(C) + 1$. When $i = 1, 2$, Transitions $r_2, r_3$ or $r_4$ can arise in $C$ except for Transitions $r_5$ and $r_7$ and their transitions does not change $q \in \{q_1, q_2\}$ to $q' \notin \{q_1, q_2\}$. Therefore, for any $r \in \{r_2, r_3, r_4\}$ a configuration $C'$ of $C \xrightarrow{r} C'$ satisfies that $\gamma_1(C) + \gamma_2(C) = \gamma_1(C') + \gamma_2(C')$. That indicates $\sum_{i=1}^{k} \gamma_i(C') \geq \sum_{i=1}^{k} \gamma_i(C)$.

2. By Lemma 3 if $\gamma_0(C) = 0$ and $\gamma_0(C') = 0$, Transition $r_5$ cannot have arisen on $C \xrightarrow{r} C'$. In arbitrary transitions except for $r_5$, $\gamma_1(C)$ decreases at most one in a transition. $\gamma_1(C)$ decreases only when a pair of agents with same state $q_1$ interact. This implies that $\gamma_1(C)$ never decreases from one to zero by any transition. \qed

Lemma 5. For any configuration $C \in \mathcal{C}$, there exists a configuration $C' \in \mathcal{L}$ and $C \xrightarrow{r} C'$.

Proof. Case 1. $\gamma_0(C) = 0$

We show that for any configuration $C \in \mathcal{C}$, there exists a configuration $C' \in \mathcal{C}$ satisfying that $C \xrightarrow{r} C'$ and $\gamma_0(C') > 0$.

Case 1.1. $\gamma_1(C) + \gamma_2(C) \leq 3$

Since $n - 4$ states $q_3, q_4, \ldots, q_{n-2}$ are assigned to at least $n - 3$ agents, there exists a pair of agents and their states are common $q_i$. When $i \geq 3$, Transitions $r_6$ or $r_7$ can arise in $C$ and exactly one of the agents changes its state from $q_i$ to $q_{i-1}$. By Lemma 4 $\gamma_i(C)(> 0)$ does not become zero by any transition and $\sum_{i=1}^{k} \gamma_i(C)$ does not decrease by any transition, thus there exists a configuration $C' \in \mathcal{C}$ satisfying that $C \xrightarrow{r} C'$ and $\gamma_1(C') + \gamma_2(C') > 3$.

Case 1.2. $\gamma_1(C) + \gamma_2(C) > 3$

Suppose Transition $r_5$ cannot have arisen on $C \xrightarrow{r} C'$, then Transitions $r_2, r_3, r_4$, or $r_7$ can arise in $C$ except for Transition $r_5$. It implies that $C$ eventually transits to a configuration $C' \in \mathcal{C}$ satisfying $\gamma_2(C') \geq 3$, thus configurations satisfying $\gamma_2(C') \geq 3$ infinitely often appear. Consider three agents $a, b, c$ with state $q_2$. A trace $(r_3; a, b), (r_3; b, c), (r_4; a, b), (r_5; c, b)$ can infinitely often arise in $C'$. Therefore, it contradicts the assumption of the global fairness, the configuration eventually transits to Case 2.

Case 2. $\gamma_0(C) > 0$

If $\gamma_0(C) > 1$, Transition $r_1$ can infinitely often have arisen by fairness condition. Hence $C$ eventually transits to a configuration $C' \in \mathcal{C}$ satisfying $\gamma_0(C') = 1$. If $\sum_{i=0}^{k} \gamma_i(C) \geq k + 3$ for $k \geq 3$, in a similar way as Case 1, $C$ eventually transits to a configuration $C'' \in \mathcal{C}$ and $\gamma_0(C'') > 2$, and $\sum_{i=0}^{k} \gamma_i(C'')$ decreases again after Transition $r_1$ arises. Since $n - 1$ states are assigned to $n$ agents and $\gamma_j(C)(> 0)$ except for $j = 1$ does not become zero by any transitions, $C$ eventually transits to $C' \in \mathcal{C}$ satisfying $\gamma_2(C') = 2$ and $\gamma_j(C') = 1$ except for $j = 2$. By Lemma 3
such a configuration $C'$ infinitely often appears, therefore Transition $r_7$ can have arisen until no edge with state $s_1$ remain $C'$. It is clear that such a configuration is included in $L$.

We obtain Theorem 1 by Lemmas 2 and 5.

4 SS-LE MPP with $(2/3)n$ Agent-States for $n$ Agents

In this section, we prove the following theorem.

**Theorem 6.** There exists a SS-LE MPP with

$$m = \left\lfloor \frac{2n}{3} \right\rfloor + 1 \equiv \begin{cases} 2n + 1 & \text{(if } n \equiv 1 \pmod{3}) \\ 2n + 2 & \text{(if } n \equiv 2 \pmod{3}) \\ 2n + 3 & \text{(if } n \equiv 0 \pmod{3}) \end{cases}$$

agent-states and 2 edge-states for $n$ agents.

We give a constructive proof. In particular, we show that Protocol $P_2$, defined as follows, is a SS-LE MPP. For simplicity of arguments, we first consider the case $n \equiv 1 \pmod{3}$.

**Protocol $P_2$ for $n \equiv 1 \pmod{3}$**

$Q = \{q_0, q_1, \ldots, q_{m-1}\}$, where $q_0$ denotes the leader state.

$S = \{s_0, s_1\}$.

$$\delta = \{$$

$r_1: (q_0, q_0, s) \mapsto (q_0, q_{m-1}, s_0)$ for $s \in S$,

$r_2: (q_{2i-1}, q_{2i-1}, s) \mapsto (q_{2i-1}, q_{2i}, s_0)$ for $s \in S$ and $i \in \{1, 2, \ldots, \frac{m-1}{2}\}$,

$r_3: (q_{2i}, q_{2i}, s_0) \mapsto (q_{2i}, q_{2i}, s_1)$ for $i \in \{1, 2, \ldots, \frac{m-1}{2}\}$,

$r_4: (q_{2i}, q_{2i}, s_1) \mapsto (q_{2i}, q_{2i-1}, s_0)$ for $i \in \{1, 2, \ldots, \frac{m-1}{2}\}$,

$r_5: (q_{2i}, q_{2i-1}, s_1) \mapsto (q_{2i}, q_{2i-2}, s_0)$,

$$((q_{2i-1}, q_{2i}, s_1) \mapsto (q_{2i}, q_{2i}, s_0), \text{symmetrically})$$ for $i \in \{1, 2, \ldots, \frac{m-1}{2}\}$,

$r_6: (q_j, q_k, s) \mapsto (q_j, q_k, s_0)$ for $j \neq k$ and $s \in S$, except for the case of $r_5$.

\}

**Remark.** Except for Transition $r_5$, the state of an agent can change only when the agent interacts with another agent in the same state.

We define a set of configurations $\mathcal{L} \subset \mathcal{C}$ by

$$\mathcal{L} \overset{\text{def}}{=} \left\{ C \in \mathcal{C} \left| \begin{array}{l} \gamma_k(C) > 0 \text{ for } k \in \{0, 1, \ldots, m-1\}, \\
\gamma_{2i-1}(C) + \gamma_{2i}(C) = 3 \text{ for } i \in \{1, 2, \ldots, \frac{m-1}{2}\}, \\
\text{both ends of an edge with state } s_1 \text{ are agents with state } q_{2i}. \end{array} \right. \right\}.$$

Let $H(i)$ for $i \in \{1, 2, \ldots, \frac{m-1}{2}\}$ denote a subconfiguration of $C \in \mathcal{L}$ consisting of three agents with states $q_{2i-1}$ or $q_{2i}$ and three edges among them. A subconfiguration $H(i)$ for $i \in \{1, 2, \ldots, \frac{m-1}{2}\}$ corresponds to $H$ in Section 3 and can
be one of three types $H_1', H_2', H_3'$ corresponding to $H_1, H_2, H_3$ in which states $q_1, q_2$ is replaced by $q_{2i-1}, q_{2i}$ respectively. Then, the edge-state $s_1$ can appear only within $H(i)$ for $i \in \{1, 2, \ldots, \frac{m-1}{2}\}$, and the number of edges with state $s_1$ in $H(i)$ is at most one.

**Lemma 7.** If configurations $C$ and $C'$ satisfy $C \in \mathcal{L}$ and $C \rightarrow C'$, then $C' \in \mathcal{L}$ and $C' \rightarrow C$ hold.

**Proof.** Transition $r_1$ cannot arise in $C \in \mathcal{L}$ by the condition $\gamma_0(C) = 1$ in $\mathcal{L}$. In a similar way as the proof of Lemma 2, $H(i)$ is closed as $H_1' \xrightarrow{r_3} H_2' \xrightarrow{r_4} H_3' \xrightarrow{r_5} H_1'$ under transitions except for $r_5$. From the condition “the both ends of an edge with state $s_1$ are $q_{2i}$”, Transition $r_5$ cannot arise in $C \in \mathcal{L}$. Thus we obtain the claim. 

Next, we show that any configuration $C \in \mathcal{C}$ eventually transits to a configuration $C' \in \mathcal{L}$, in Lemma 10. To show Lemma 10 we show Lemmas 8 and 9.

**Lemma 8.** If a configuration $C$ satisfies $\gamma_{2i-1}(C) + \gamma_{2i}(C) \geq 4$, then Transition $r_5$ can eventually arise.

**Proof.** Let $C'$ be an arbitrary configuration which $C$ eventually transits to, and suppose Transition $r_5$ cannot have arisen on $C \rightarrow C'$. Transitions $r_2, r_3, r_4$ or $r_6$ can arise except for Transitions $r_1, r_5$. Note that Transitions $r_3$ and $r_6$ do not change any agent-states. Exactly one of two agents with a common state changes its state from $q_{2i-1}$ to $q_{2i}$ by Transition $r_2$, and also changes its state from $q_{2i}$ to $q_{2i-1}$ by Transition $r_4$. This implies that $C$ eventually transits to a configuration $C' \in \mathcal{C}$ satisfying $\gamma_{2i}(C') \geq 3$. Thus some configurations satisfying $\gamma_{2i}(C') \geq 3$ infinitely often appear. Consider three agents with state $q_{2i}$ as $a, b, c$. A trace $(r_3; a, b), (r_3; b, c), (r_4; a, b), (r_5; c, b)$ can infinitely often appear in $C'$ see also Fig. 2 $D_k(i)$ for $k = 1, 2, \ldots, 5$ denote a subconfiguration of $C \in \mathcal{C}$ consisting of four agents whose states are $q_{2i}, q_{2i-1}$ or $q_{2i-2}$. Since it contradicts the assumption of the global fairness, we obtain the claim. 

**Lemma 9.** If a configuration $C \in \mathcal{C}$ satisfies $\gamma_0(C) = 0$ and a configuration $C' \in \mathcal{C}$ which $C$ eventually transits to satisfies $\gamma_0(C') = 0$, then $\sum_{i=1}^{2k} \gamma_i(C') \geq \sum_{i=1}^{2k} \gamma_i(C)$ for all $k \in \{1, 2, \ldots, \frac{m-1}{2}\}$.

**Proof.** If $\gamma_0(C) = 0$ and $\gamma_0(C') = 0$ hold, Transitions $r_1, r_5$ cannot have arisen on $C \rightarrow C'$. Note that Transition $r_6$ does not change any agent-state. Since $m-1$ states are assigned to $\frac{3}{2}(m-1)+1$ agents, there exists a subconfiguration $H(j)$ of the configuration $C$ satisfying $\gamma_{2j-1}(C) + \gamma_{2j}(C) > 3$ for $j \in \{1, 2, \ldots, \frac{m-1}{2}\}$. By Lemma 8 an agent in $C$ eventually changes its state from $q_{2j-1}$ to $q_{2j-2}$, and let $C' \in \mathcal{C}$ denote the configuration, then $\sum_{i=1}^{2j} \gamma_i(C') = \sum_{i=1}^{2j} \gamma_i(C)$ and $\gamma_{2j-2}(C') = \gamma_{2j-2}(C) + 1$ hold. Let $C''$ be an arbitrary configuration which $C$ eventually transits to and suppose Transitions $r_1, r_5$ cannot have arisen on $C \rightarrow C'$. $\gamma_{2j-1}(C') + \gamma_{2j}(C') = \gamma_{2j-1}(C) + \gamma_{2j}(C)$ holds for $C \rightarrow C''$. Therefore, $\sum_{i=1}^{2k} \gamma_i(C') \geq \sum_{i=1}^{2k} \gamma_i(C)$ for all $k \in \{1, 2, \ldots, \frac{m-1}{2}\}$.
Fig. 2. A trace of a subconfiguration of $C$ which satisfies $\gamma_{2i-1}(C) + \gamma_{2i}(C) \geq 4$

Fig. 3. States transition diagram of subconfigurations of $C$ which satisfy $\gamma_{2i-1}(C) + \gamma_{2i}(C) \geq 4$
**Lemma 10.** For any configuration $C \in \mathcal{C}$, there exists a configuration $C' \in \mathcal{L}$ and $C \xrightarrow{\ast} C'$.

**Proof.** Case 1. $\gamma_0(C) = 0$

We show that any configuration $C \in \mathcal{C}$ eventually transits to a configuration $C' \in \mathcal{C}$ satisfying $\gamma_0(C') > 0$. Since $m - 1$ states are assigned to $\frac{3}{2}(m - 1) + 1$ agents, there exists a subconfiguration $H(j)$ of the configuration $C$ satisfying $\gamma_2(j - 1(H(j))) + \gamma_2(H(j)) > 3$ for $j \in \{1, 2, \ldots, \frac{m}{2}\}$. By Lemma 9 $C$ eventually transits to a configuration $C' \in \mathcal{C}$ satisfying $\gamma_1(C') + \gamma_2(C') > 3$, and then eventually transits to a configuration $C'' \in \mathcal{C}$ satisfying $\gamma_0(C'') > 0$ in a similar way as the proof of Lemma 5.

Case 2. $\gamma_0(C) > 0$

If $\gamma_0(C) > 0$, Transition $r_1$ can infinitely often have arisen under fairness condition. The configuration $C$ eventually transits to a configuration $C' \in \mathcal{C}$ satisfying $\gamma_0(C') \geq 2$, and $\gamma_0(C) + \sum_{i=1}^{2k} \gamma_i(C)$ for $k \in \{1, 2, \ldots, \frac{m}{2}\}$ decreases again after Transition $r_1$ arises. Thus, $C$ infinitely often transits to a configuration $C'' \in \mathcal{C}$ satisfying $\gamma_0(C'') = 1$ and $\gamma_2i-1(C'') + \gamma_2i(C'') = 3$ for every $i \in \{1, 2, \ldots, \frac{m}{2}\}$.

Since $C''$ infinitely often appears, Transition $r_6$ can have arisen until no edge with state $s_1$ remain $C''$. It is clear that such a configuration is included in $\mathcal{L}$. □

By Lemmas 7 and 10 we obtain a SS-LE MPP with $m$ agent-states and 2 edge-states for $n$ agents in case of $n \equiv 1 \pmod{3}$.

Next we give Protocol $P_2$ for $n \equiv 2 \pmod{3}$ agents. We define Protocol $P_2$ for $n \equiv 2 \pmod{3}$ by adding

$$r_7 : (q_{m-1}, q_{m-1}, s) \mapsto (q_{m-1}, q_{m-2}, s_0)$$

to the transition function $\delta$ of Protocol $P_2$ for $n \equiv 1 \pmod{3}$, and appropriately modify the domain of $i$ as $\{1, 2, \ldots, \frac{m-2}{2}\}$ in Protocol $P_2$ for $n \equiv 1 \pmod{3}$.

We define a set of configurations $\mathcal{L} \subset \mathcal{C}$ by

$$\mathcal{L} \overset{\text{def}}{=} \left\{ C \in \mathcal{C} \mid \begin{array}{l}
\gamma_k(C) > 0 \text{ for } k \in \{0, 1, \ldots, m-1\}, \\
\gamma_{2i-1}(C) + \gamma_{2i}(C) = 3 \text{ for } i \in \{1, 2, \ldots, \frac{m-2}{2}\}, \\
\text{both ends of an edge with state } s_1 \text{ are agents with state } q_{2i}.
\end{array} \right\}.$$

Let $H(i)$ for $i \in \{1, 2, \ldots, \frac{m-2}{2}\}$ denote a subconfiguration of $C \in \mathcal{L}$ consisting of three agents with states $q_{2i-1}$ or $q_{2i}$, and three edges among them.

**Lemma 11.** If configurations $C'$ and $C''$ satisfy $C \in \mathcal{L}$ and $C \xrightarrow{\ast} C'$, then $C' \in \mathcal{L}$ and $C' \xrightarrow{\ast} C''$ hold.

**Proof.** By the conditions $\gamma_k(C) > 0$ for $k \in \{0, 1, \ldots, m-1\}$ and $\gamma_{2i-1}(C) + \gamma_{2i}(C) = 3$ for $i \in \{1, 2, \ldots, \frac{m-2}{2}\}$, $\gamma_0(C) = \gamma_{m-1}(C) = 1$ and Transitions $r_1, r_7$ cannot arise in $C$. Thus we obtain the claim in a similar way as the proof of Lemma 7. □

Next, we prove that any configuration $C \in \mathcal{C}$ eventually transits to a configuration $C' \in \mathcal{L}$ (Lemma 12). If $\gamma_0(C) = 0$, since $m - 1$ states are assigned to
\[\frac{3}{2}(m - 2) + 2\] agents, there exists a subconfiguration \(H(i)\) of the configuration \(C\) satisfying \(\gamma_{2i-1}(H(i)) + \gamma_{2i}(H(i)) \geq 3\) for \(i \in \{1, 2, \ldots, \frac{m-2}{2}\}\), or there exists a pair of agents with the common state \(q_{m-1}\) in \(C\). Transition \(r_7\) arises in a configuration \(C \in \mathcal{C}\) and exactly one of the agents changes its state from \(q_{m-1}\) to \(q_{m-2}\). Using (a type of) Lemmas 8 and 9, we can show the following.

**Lemma 12.** For any configuration \(C \in \mathcal{C}\), there exists a configuration \(C' \in \mathcal{L}\) and \(C \xrightarrow{*} C'\).

**Proof.**

**Case 1.** \(\gamma_0(C) = 0\)

We show that any configuration \(C \in \mathcal{C}\) eventually transits to a configuration \(C' \in \mathcal{C}\) satisfying \(\gamma_0(C') > 0\). Since \(m - 1\) states are assigned to \(\frac{3}{2}(m - 2) + 2\) agents, there exists a subconfiguration \(H(i)\) of the configuration \(C\) satisfying \(\gamma_{2i-1}(H(i)) + \gamma_{2i}(H(i)) \geq 3\) for \(i \in \{1, 2, \ldots, \frac{m-2}{2}\}\) or there exists a pair of agents and their states are common \(q_{m-1}\). In the former case, \(C\) eventually transits to a configuration \(C' \in \mathcal{C}\) satisfying \(\gamma_1(C') + \gamma_2(C') > 3\) by Lemma 9.

In the latter case, Transition \(r_7\) can arise in \(C\) and exactly one of two agents with common \(q_{m-1}\) changes its state to \(q_{m-2}\). Therefore, \(C\) eventually transits to a configuration \(C' \in \mathcal{C}\) satisfying \(\gamma_1(C') + \gamma_2(C') > 3\), and then eventually transits to a configuration \(C'' \in \mathcal{C}\) satisfying \(\gamma_0(C'') > 0\).

**Case 2.** \(\gamma_0(C) > 0\)

Shown in a similar way as Case 2 of the proof of Lemma 10. \(\square\)

Finally, we present our SS-LE MPP for \(n\) agents in case of \(n \equiv 0\) (mod 3). We define Protocol \(P_2\) for \(n \equiv 0\) (mod 3) by adding

\[r_8: (q_{m-2}, q_{m-2}, s) \mapsto (q_{m-2}, q_{m-3}, s_0)\]

to the transition function \(\delta\) of Protocol \(P_2\) for \(n \equiv 2\) (mod 3) and appropriately replace the domain of \(i\) as \(\{1, 2, \ldots, \frac{m-3}{2}\}\) in Protocol \(P_2\) for \(n \equiv 2\) (mod 3).

Let \(H(i)\) for \(i \in \{1, 2, \ldots, \frac{m-3}{2}\}\) denote a subconfiguration of \(C \in \mathcal{L}\) consisting of three agents with states \(q_{2i-1}\) or \(q_{2i}\) and three edges among them.

It is not difficult to see \(\mathcal{L}\), appropriately modifying the domain of \(i\), is the set of legal configurations of the case, in a similar way as the case of \(n \equiv 2\) (mod 3).

Now we obtain Theorem 6.

## 5 Lower Bounds for SS-LE MPP

In this section, we give two lower bounds of the number of agent-states.

**Theorem 13.** For general \(n\) agents, any SS-LE MPP with a constant number of agent-states and a constant number of edge-states does not exist.

**Proof.** Suppose \(P(n)\) is a SS-LE MPP with a constant agent-states and a constant edge-states for \(n\) agents. We show that \(P(n)\) cannot be a SS-LE MPP for \(n'\) agents where \(n' \neq n\). Without loss of generality we may assume that \(n > n'\). Let \(C\) be a legal configuration of \(P(n)\) and let \(D\) be a subconfiguration of \(C\) where
$D$ consists of $n'$ non-leader agents. Then $P(n)$ cannot create leader states by any interactions within $D$, since $D$ is a subconfiguration of a legal configuration $C$ of $P(n)$. This means that $P(n)$ is not a SS-LE MPP for $n'$ agents, because $D$ cannot reach at a legal configuration for $n'$ agents.

When possible agent-states and edge-states are finite, the number of components included in the transition function is also finite. In fact, for $c$ agent-states and $d$ edge-states, the number of possible protocols is (at most) $(c^2d)^{c^2d}$. Since the number of agents $n$ can be fairly larger than them, we obtain the claim. □

**Theorem 14.** If the number of agent-states is 2, then no SS-LE MPP exists for $n(>2)$ agents, even when the number of edge-states is infinitely large.

**Proof.** With an assumption that a legal configuration exists and we derive a contradiction. When $n > 2$, the number of an agent with a leader state $q_0$ included in a legal configuration $L$ is one and the number of agents with non-leader state $q_1$ is at least two. Therefore it is clear that the state of agents with state $q_1$ in a configuration $C \in L$ does not change. Let $C' \in C$ be a configuration which is constructed by the 3 tuple of two agents and an edge among them which are included in $C$, then $C'$ cannot transit to a configuration which includes a agent-state $q_0$. This contradicts any initial configuration eventually transits to a configuration including a leader state. □

### 6 Conclusion

We gave a SS-LE MPP with roughly $(2/3)n$ agent-states and two edge-states for $n$ agents, while we showed that there is no SS-LE MPP with any constant agent-states and any constant edge-states for general $n$ agents. We conjecture that our upper bound is almost tight. A future work is to analyze SS-LE MPP with a constant edge-states. Analyses on other interaction graphs may be another future work.

### References


